PROF. DR. AHMET YÜCESAN

GEOMETRY, ALGORITHMS AND VARIATIONS:

MODERN MATHEMATICAL THEORIES

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PROF. DR. AHMET YÜCESAN

Prof. Dr. Ahmet Yücesan is a faculty member at the Department of Mathematics, Faculty of Engineering and Natural Sciences, Süleyman Demirel University. Prof. Dr. Yücesan, who completed his undergraduate education at the Department of Mathematics, Eskişehir Osmangazi University, received his master's and doctoral degrees at the Department of Mathematics, Graduate School of Natural and Applied Sciences, Süleyman Demirel University. Throughout his academic career, he has undertaken many important researches in the fields of differential geometry, theory of curves and surfaces, manifold theory, Minkowski space and variational problems.

His scientific work encompasses a wide spectrum of both theoretical and applied mathematics. He is particularly recognized for his studies on the geometric properties of curves in Euclidean and Minkowski space, focusing on elastic curves, elastic strips, Bézier curves, submanifolds and manifolds with semi-symmetric connection. His contributions include numerous articles published in national and international journals, presentations at conferences, and manager in several scientific projects.

In addition to his academic research, Prof. Dr. Yücesan is dedicated to educating students at undergraduate and graduate levels, playing a crucial role in nurturing young researchers. His passion for advancing mathematical thinking and sharing knowledge with students and colleagues continues to contribute to the progression of science.

Married with two children, Prof. Dr. Yücesan enjoys nature walks and swimming during his free time.

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Preface

The pursuit of knowledge in mathematics consistently yields profound insights into the theoretical frameworks that underpin our understanding of the structure of the natural world and complex structures. This book represents a collection of work that explores the boundaries of mathematical reasoning, algorithmic design, and geometric analysis, showcasing the diverse applications and innovative methodologies that define modern research.

From the discovery of unit fractions and the development of mathematical generalizations to the complex behavior of helix, rectifying, hyperquadrical, and hyperelastic curves in Galilean 3-space and Minkowski geometry, this collection serves as a testament to the dynamic interplay between theory and practice. Each chapter exemplifies the creativity and rigor of its contributors, providing both foundational knowledge and new perspectives for readers interested in the mathematical sciences and their applications.

The chapters presented here also highlight the critical role of mathematical modeling and algorithm development in addressing real-world problems such as optimizing spatial representations in service delivery. These contributions highlight not only the elegance of mathematics as a discipline but also its practical importance in solving everyday problems and advancing technological innovation.

I sincerely hope that this book will inspire researchers, educators, and students to further explore the depths and limitless applications of mathematical theory. The collective efforts of the authors and their commitment to academic excellence have culminated in a resource that will undoubtedly enrich its readers' understanding and appreciation of mathematics.

I express my gratitude to all contributors for their extraordinary work and dedication, and I am confident that readers will find this collection as enlightening and engaging as it was for those who collaborated in its creation.

CONTENTS

CHAPTER 1

A STUDY ON THE NATURAL AND CONJUGATE MATES OF NON-NULL SPACE CURVES

Ahmet YÜCESAN[1]

Adile Gökçe ÇINAR[2]

In this chapter, we first derive the position vector of the natural mate of a non-null curve in Minkowski 3 −space by using the Frenet apparatus of the curve and a function that includes the distance function of the natural mate and its derivative. Using the position vector of the natural mate of a non-null curve, we establish the necessary and sufficient conditions for the non-null curve to be a Bertrand curve. As an example, we give spacelike natural mate of spacelike Salkowski curve with a spacelike principal normal. We then obtain a new characterization of the general helix. Finally, we construct the position vector of the conjugate mate of a non-null space curve with the aid of the Frenet apparatus of the curve and a function containing the distance function of the conjugate mate, its derivatives and the torsion of the curve. We present a result for the case where the conjugate mate of a non-null curve is a hyperquadrical curve or a rectifying curve.

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1. Introduction

Differential geometry of curves is one of the most interesting topics in both Euclidean $3-$space and Minkowski $3-$space. In particular, some special curves such as general helix, spherical curves (hyperquadrical curves in Minkowski $3-$space), rectifying curves and Bertrand curves are of more interest to study. On the other hand, some mathematicians have studied various associated curves of a given curve. These associated curves can characterize an original curve and explain its behavior. Associated curves are given as integral curves of some vector fields generated by the elements of the Frenet frame along the curve (Choi and Kim, 2012; Choi et al., 2012; Deshmukh et al., 2018b; Kelleci, 2019; Kelleci, 2020; Chen, 2023; Alghanemi et al., 2023).

With this motivation, we study the natural mate (principal direction curve or integral curve of the principal normal vector field) and the conjugate mate (binormal direction curve or integral curve of the binormal vector field) of non-null curves in Minkowski $3-$space and characterize the special curves mentioned above.

We now begin by recalling Minkowski $3-$space (or the $3-$dimensional Lorentzian space), denoted by R_1^3. The first step is to define a new inner product on Euclidean $3-$space R^3, called the Lorentzian inner product. This leads to some new concepts, and we express these concepts from (O'Neill, 1983; Birman and Nomizu, 1984; Ratcliffe, 1994; Barros et al., 2001; López, 2014). The Minkowski $3-$space R_1^3 is a real linear space with indefinite inner product given by

$$< u, v >= -u_1v_1 + u_2v_2 + u_3v_3 \tag{1.1}$$

for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in R^3. Then, the norm of a vector u in R_1^3 is defined as the real number $\|u\| = \sqrt{|< u, u >|}$. The Lorentzian vector product of u and v is given by

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

A vector in R_1^3 has three different causal characters as spacelike, timelike and null (lightlike). If $< u,u >> 0$ (or $u = 0$), $< u,u >< 0$, $< u,u >= 0$ (and $u \neq 0$) then a vector u is named to be spacelike, timelike or null, respectively. A plane in Minkowski $3-$space is called spacelike, timelike or null if the restriction on this plane of the inner product defined by (1.1) is positive definite, indefinite non-degenerate or degenerate, respectively.

Let us assume that u and v are two non-null vectors in R_1^3. Then, depending on whether u and v are spacelike or timelike, the notions of Lorentzian timelike or spacelike angles are given as follows:

i) If u and v are spacelike vectors that span a timelike vector subspace, a unique positive real number θ exists, referred to as the Lorentzian timelike angle between u and v, satisfying the condition that

$$| < u,v > | = \|u\|\|v\|cosh\theta,$$

ii) If u is a spacelike vector and v is a positive timelike vector, then there is a unique non-negative real number θ is known the Lorentzian timelike angle between u and v such that

$$| < u,v > | = \|u\|\|v\|sinh\theta,$$

iii) If u and v are positive (negative) timelike vectors, then there is a unique non-negative real number θ called the Lorentzian timelike angle between u and v, satisfying the condition that

$$< u,v >= \|u\|\|v\|cosh\theta,$$

iv) If u and v are spacelike vectors that span a spacelike vector subspace, a unique real number θ, $0 \leq \theta \leq \pi$, exists, referred to as the Lorentzian spacelike angle between u and v such that

$$< u,v >= \|u\|\|v\|cos\theta.$$

A smooth curve is a differentiable map $\gamma: I \subset R \rightarrow R_1^3$, where I is an open interval. We know that the causal character of the curve is determined by the causal character of its velocity. Therefore, a curve γ is said to spacelike, timelike or null if its velocity vector is spacelike, timelike or null vector, respectively. Lastly, a surface is named non-degenerate (or degenerate) if induced metric on its tangent plane is non degenerate (or degenerate). As known, the most familiar non-degenerate surfaces are pseudo-sphere of radius $r > 0$

$$S_1^2 = \{p \in R_1^3 \mid \, < p, p >= r^2, r > 0\}$$

and the pseudo-hyperbolic space of radius $r > 0$

$$H_0^2 = \{p \in R_1^3 \mid \, < p, p >= -r^2, r > 0\}.$$

These surfaces are called the hyperquadrics of R_1^3. A curve lies on hyperquadrics is called hyperquadrical curve. While there are three different causal characters of a curve on the pseudo-sphere, there are only spacelike curves on the pseudo-hyperbolic space.

As is known, a curve is studied in Minkowski $3-$space R_1^3 , as in Euclidean $3-$space R^3, by assigning at each point a certain frame. If the curve can be continuously differentiable at least three times, choosing frame along the curve is usually the Frenet frame $\{T, N, B\}$. Then the ratio of changes of the three vectors T, N and B along the curve is expressed in terms of the vectors themselves by the celebrated Frenet formulas. With this understanding, the theory of curves in R_1^3 is only a consequence of these fundamental formulas. The formulas are expressed as follows: Let $\gamma: I \rightarrow R_1^3$, be a regular and non-null curve. The arclength parameter of γ is determined by s such that $\|\gamma'(s)\| = 1,\ \gamma'(s) = \frac{d\gamma}{ds}$. At a point $\gamma(s)$ of γ, let $T(s) = \gamma'(s)$ denote the unit tangent vector, $N(s)$ the unit principle normal vector and $\varepsilon_2 B(s) = T(s) \times N(s)$ the unit binormal vector of γ, where $\varepsilon_2 =< B(s), B(s) >= \pm 1$. The Frenet formulas of the frame $\{T(s), N(s), B(s)\}$ are given by

$$\begin{aligned} T'(s) &= \varepsilon_1 \kappa(s) N(s) \\ N'(s) &= -\varepsilon_0\, \kappa(s) T(s) + \varepsilon_2 \tau(s) B(s) \\ B'(s) &= -\varepsilon_1\, \tau(s) N(s), \end{aligned} \tag{1.2}$$

where $\varepsilon_0 =< T(s), T(s) >= \pm 1$, $\varepsilon_1 =< N(s), N(s) >= \pm 1$, $\kappa(s)$ is the curvature and $\tau(s)$ is the torsion of γ at s.

After introducing Minkowski $3-$space, let us state the definitions and characterizations of the special curves we mentioned above:

Definition 1.1 A general helix $\gamma: I \subset R \to R_1^3$ is a regular curve parametrized by arc length (or by the pseudo arc length if γ is null) such that there exists a non-zero constant vector $U \in R_1^3$ with the property that the function $< T, U >$ is a constant. Any line parallel this direction U is called the axis of the general helix (Barros et al., 2001; Lopez, 2014).

Theorem 1.1 Let γ be a non-null curve with non-null principal normal in Minkowski $3-$space R_1^3. Then, γ is a general helix if and only if τ/κ is constant (Barros et al., 2001; Lopez, 2014).

Definition 1.2 Let $\gamma = \gamma(s)$ be a regular curve in Minkowski $3-$space R_1^3. Then, γ is called a rectifying curve if its position vector always lies in its rectifying plane at each point (Kim et al., 1993; Chen, 2003; İlarslan et al., 2003).

Theorem 1.2 Let $\gamma = \gamma(s)$ be a unit speed rectifying curve in R_1^3 with spacelike or timelike rectifying plane, the curvature $\kappa(s) > 0$. Then the following statements hold:

i) The distance function $d = \|\gamma(s)\|$ satisfies

$$d^2 = |\varepsilon_0 s^2 + c_1 s + c_2|, \tag{1.3}$$

for some $c_1, c_2 \in R$ with $c_2 \neq 0$.

ii) The tangential component of the position vector of γ is given by

$$< \gamma, T >= \varepsilon_0 s + c,$$

where $c \in R$

iii) The normal component γ^N of the position vector of the curve has a constant length and the distance function d is non-constant

iv) The torsion $\tau(s) \neq 0$ and the binormal component of the position vector of the curve is constant, i.e. $< \gamma, B >$ is a constant (İlarslan et al., 2003).

Theorem 1.3 Let $\gamma: I \subset R \to R_1^3$ be a unit speed curve with a spacelike or a timelike rectifying plane and with the curvature $\kappa(s) > 0$. Then γ is congruent to a rectifying curve if and only if the ratio of torsion and curvature of the curve is a non-constant linear function in arclength function s, i.e., $\frac{\tau(s)}{\kappa(s)} = c_1 s + c_2$ for some $c_1, c_2 \in R$ with $c_2 \neq 0$ (İlarslan et al., 2003).

Definition 1.3 A regular curve γ is said to be a hyperquadrical curve if it lies on hyperquadrics (O'Neill, 1983).

Theorem 1.4 Let $\gamma = \gamma(s)$ be a unit speed curve in R_1^3 with $\kappa > 0$ and $\tau \neq 0$. Then γ is a hyperquadrical curve, i.e. for it to lie on a pseudo-sphere or pseudo-hyperbolic space, if and only if

$$\varepsilon_1 \left(\frac{1}{\tau} \left(\frac{1}{\kappa} \right)' \right)' + \varepsilon_2 \left(\frac{\tau}{\kappa} \right) = 0$$

(see Struik, 1961; Pekmen and Paşalı, 1999; Petrovic-Torgasev and Sucurovic, 2000a,b; Petrovic-Torgasev and Sucurovic, 2001; İlarslan et al., 2003).

Definition 1.4 A non-null curve γ with non-zero curvature is said to be a Bertrand curve if there exists another immersed non-null curve $\beta = \beta(\sigma): J \subset R \rightarrow R_1^3$, $\beta \neq \pm\gamma$, and a one-to-one correspondence between γ and β (i.e. a map $s \in I \rightarrow \sigma(s) \in J$), such that both curves have common principal normal at corresponding points. β is called the Bertrand mate of γ; the curves γ and β are said that a pair of non-null Bertrand curves (Lucas and Ortega-Yagües, 2013).

Theorem 1.5 A non-null curve γ with $\kappa > 0$ and $\tau \neq 0$ in R_1^3 is a Bertrand curve if and only if there exist two constants $\lambda \neq 0$ and μ such that

$$\lambda\kappa + \mu\tau = 1. \qquad (1.4)$$

(Balgetir et al., 2004; Lucas and Ortega-Yagües, 2013).

Definition 1.5 The curve $\gamma: I \subset R \rightarrow R_1^3$ given by

$$\gamma(t) = \frac{n}{4m}\Big(2cosh(t) - \frac{1+n}{1-2n}cosh\big((1-2n)t\big) - \frac{1-n}{1+2n}cosh\big((1+2n)t\big), \qquad (1.5)$$

$$2cosh(t) - \frac{1+n}{1-2n}sinh\big((1-2n)t\big) - \frac{1-n}{1+2n}sinh\big((1+2n)t\big), \frac{1}{m}\cosh(2nt)\Big)$$

with $n = \frac{m}{\sqrt{m^2-1}}$ for any $m \in R$ with $m > 1$ is called a spacelike Salkowski curve with a spacelike principal normal (Ali, 2009).

Definition 1.6 Let $\gamma: I \subset R \rightarrow R_1^3$ be a unit speed non-null curve with Frenet frame $\{T, N, B\}$. Then a unit speed curve β given by $\beta(s) = \int N ds$ is called the natural mate (or principal direction curve) of γ (Choi and Kim, 2012; Choi et al., 2012; Deshmukh et al., 2018b; Kelleci, 2020).

Definition 1.7 Let $\gamma: I \subset R \rightarrow R_1^3$ be a unit speed with Frenet frame $\{T, N, B\}$. Then a unit speed curve α given by $\alpha(s) = \int B ds$ is called the conjugate mate (or binormal direction curve) of γ (Choi and Kim, 2012; Choi et al., 2012; Deshmukh et al., 2018b; Kelleci, 2019).

2. Natural and Conjugate Mates of Non-null Space Curves

First, we give the position vector of the natural mate of a non-null curve. With the help of the position vector of the curve, we study some relationship between given a non-null curve and its natural mate. Then, we are similarly interested in the conjugate mate of the curve.

Let $\gamma: I \subset R \to R_1^3$ be a unit speed curve with Frenet apparatus $\{\kappa, \tau, T, N, B\}$ in Minkowski 3 $-$space R_1^3. Then the natural mate β of γ is

$$\beta(s) = \int N ds = \varepsilon_0 gT + \varepsilon_1 hN + \varepsilon_2 lB, \tag{2.1}$$

where g, h and l are differentiable functions. We begin with the derivative of (2.1) with respect to s. Then we find

$$N = \varepsilon_0(g' - \varepsilon_1 \kappa h)T + \varepsilon_1(\varepsilon_0 g\kappa + h' - \varepsilon_2 l\tau)N + \varepsilon_2(\varepsilon_1 h\tau + l')B. \tag{2.2}$$

From (2.2), we obtain the following system of equations:

$$\begin{cases} g' - \varepsilon_1 \kappa h = 0 \\ \varepsilon_0 g\kappa + h' - \varepsilon_2 l\tau = \varepsilon_1 \\ \varepsilon_1 h\tau + l' = 0. \end{cases} \tag{2.3}$$

So we have

$$\begin{aligned} g &= \varepsilon_1 \int (\kappa h) ds, \\ l &= -\varepsilon_1 \int (\tau h) ds. \end{aligned} \tag{2.4}$$

On the other hand, the distance squared function, d^2 of β is given by

$$\delta d^2 = \varepsilon_0 g^2 + \varepsilon_1 h^2 + \varepsilon_2 l^2, \tag{2.5}$$

where $\delta = \frac{<\beta,\beta>}{|<\beta,\beta>|} = \pm 1$ (see Deshmukh et al., 2018a; Engin and Yücesan, 2020). By differentiating (2.5) with respect to s and using (2.3) and (2.4), we get

$$h = \delta d d'.$$

Then we can give the following theorem.

Theorem 2.1 Let $\gamma = \gamma(s)$ be a unit speed curve with Frenet apparatus $\{\kappa, \tau, T, N, B\}$ in Minkowski 3 −space R_1^3. Then the natural mate β of γ is given by

$$\beta(s) = \varepsilon_0\varepsilon_1\int (\kappa h)ds T + \varepsilon_1 hN - \varepsilon_1\varepsilon_2\int (\tau h)ds B, \qquad (2.6)$$

where $h = \delta dd'$, d is the distance function of β and $\delta = \frac{<\beta,\beta>}{|<\beta,\beta>|} = \pm 1$.

Now, using Theorem 2.1, we give a theorem concerning the Bertrand curve, the hyperquadrical curve and the rectifying plane. Then, we consider the spacelike Salkowski curve with spacelike principal normal and its natural mate lying on the pseudo-sphere S_1^2.

Theorem 2.2 Let $\gamma : I \to R_1^3$ be a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$ and β be its natural mate. Then the following statements hold:

i) γ is a non-null Bertrand curve,

ii) β is a hyperquadrical curve,

iii) β lies in the rectifying plane of γ.

Proof. Let $\gamma = \gamma(s)$ be a unit speed curve with $\kappa > 0$ and $\tau \neq 0$ and β be its natural mate. From Theorem 2.1, β is given by (2.6) that is

$$\beta(s) = \varepsilon_0\varepsilon_1\int (\kappa h)ds T + \varepsilon_1 hN - \varepsilon_1\varepsilon_2\int (\tau h)ds B,$$

where $h = \delta dd'$ and d is the distance function of β. Then the natural mate β of the curve γ is a hyperquadrical curve (i.e. $d = \pm r = const > 0$ and $dd' = 0$) if and only if

$$\beta(s) = \varepsilon_0\varepsilon_1 c_1 T - \varepsilon_1\varepsilon_2 c_2 B, \qquad (2.7)$$

where c_1 and c_2 are non-zero constants and it is understand that β lies in the rectifying plane of γ. Considering Definition 1.6 and taking derivative of both sides (2.7) with respect to s, we find

$$\varepsilon_0 c_1\kappa + \varepsilon_2 c_2\tau = 1$$

if and only if γ is a non-null Bertrand curve.

As an application of Theorem 2.2, we give a simple result about Salkowski curves before giving that a spacelike Salkowski curve with a spacelike principal normal and its Bertrand mate lying on the pseudo-sphere S_1^2.

Corollary 2.1 If γ is a non-null Salkowski curve, then its natural mate β is given by

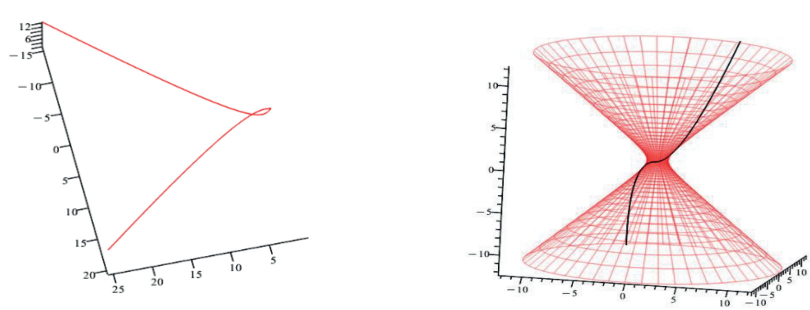
$$\beta = \frac{\varepsilon_1}{\kappa} T. \tag{2.8}$$

Example 2.1 Let $\gamma: I \subset R \to R_1^3$, be spacelike Salkowski curve with a spacelike principal normal given by (1.5). This curve has $\kappa(t) = 1$ and $\tau(t) = tanh(nt)$. From (2.8), the natural mate β of γ is

$$\beta = (ncosh(t)sinh(nt) - sinh(t)cosh(nt),$$

$$nsinh(t)sinh(nt) - cosh(t)cosh(nt), \frac{n}{m} sinh(nt)),$$

where $n = \frac{m}{\sqrt{m^2-1}}$ for any $m \in R$ with $m > 1$ and β lies on a unit pseudo-sphere S_1^2. When $m = 15$, the spacelike Salkowski curve with a spacelike principal normal and its natural mate are drawn as shown in Figure 2.1(a) and Figure 2.1(b), respectively.



(a) The curve γ (b) The black curve is β lying on the pseudo-sphere S_1^2.

Figure 2.1 When $m = 15$, the spacelike Salkowski curve γ with a spacelike principal normal and its natural mate β.

Theorem 2.3 Let $\gamma : I \to R_1^3$ be a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$. Then, γ is a general helix if and only if there exists a fixed direction orthogonal to its natural mate β.

Proof. Let $\gamma : I \to R_1^3$ be a unit speed curve with the curvature $\kappa > 0$, the torsion $\tau \neq 0$ and β be its natural mate. Suppose that γ is a general helix, then there exists a fixed direction makes a constant angle with its tangent. Let U be a unit constant vector lies on that direction, then $< T, U >= \phi(\theta) = constant$ and $< B, U >= \eta(\theta) = constant$, where θ is the fixed angle between T and U. Also, ϕ and η are constant functions satisfying

$$\varepsilon_0 \phi^2(\theta) + \varepsilon_2 \eta^2(\theta) = \varepsilon,$$

where $\varepsilon =< U, U >= \pm 1$. Considering into Theorem 2.1, we have

$$< \beta, U >= \varepsilon_0 \varepsilon_1 \, \phi(\theta) \int (\kappa h) ds - \varepsilon_1 \varepsilon_2 \, \eta(\theta) \int (\tau h) ds.$$

Since γ is a general helix, then

$$\varepsilon_0 \varepsilon_1 \, \phi(\theta) \int (\kappa h) ds - \varepsilon_1 \varepsilon_2 \, \eta(\theta) \int (\tau h) ds = 0$$

imply that $< \beta, U >= 0$ that is U is ortogonal to β.

Conversely, suppose that there is a fixed direction orthogonal to β and let U be a unit constant vector lies on this direction. Then $< \beta, U >= 0$. Taking derivative in both sides of this equation with respect to s, we obtain $< N, U >= 0$ and using the Frenet formulas (1.2) we get $< T, U >= constant$. Therefore γ is a general helix.

Similarly, we give the position vector of the conjugate mate of a non-null curve in Minkowski $3-$space.

Theorem 2.4 Let $\gamma = \gamma(s)$ be a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$ in Minkowski $3-$space R_1^3. Then the conjugate mate α of γ is given by

$$\alpha(s) = \varepsilon_0\varepsilon_1 \int (\kappa h_1)ds\, T + \varepsilon_1 h_1 N + (s - \varepsilon_1\varepsilon_2 \int (\tau h_1)ds)B \qquad (2.9)$$

where $h_1 = \frac{\varepsilon_1}{\tau}(\varepsilon_2 - \delta_1({d_1'}^2 + d_1 d_1''))$, d_1 is the distance function of α and $\delta_1 = \frac{<\alpha,\alpha>}{|<\alpha,\alpha>|} = \pm 1$.

Proof. Suppose that $\gamma = \gamma(s)$ is a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$ and α is the conjugate mate of α. Then we can write

$$\alpha(s) = \int B ds = \varepsilon_0 g_1 T + \varepsilon_1 h_1 N + \varepsilon_2 l_1 B, \qquad (2.10)$$

where g_1, h_1 and l_1 are differentiable functions. Differentiation of (2.10) with respect to s and using the Frenet formulas (1.2), we obtain

$$\begin{cases} g_1' - \varepsilon_1 \kappa h_1 = 0 \\ \varepsilon_1 h_1' + \varepsilon_0\varepsilon_1 \kappa g_1 - \varepsilon_1\varepsilon_2 l_1 \tau = 0 \\ \varepsilon_1 h_1 \tau + l_1' = \varepsilon_2. \end{cases} \qquad (2.11)$$

From (2.11), we have

$$\begin{aligned} g_1 &= \varepsilon_1 \int \kappa h_1 ds, \\ l_1 &= \varepsilon_2 s - \varepsilon_1 \int \tau h_1 ds\,. \end{aligned} \qquad (2.12)$$

If we define the distance squared function of α as d_1^2 then it is written as

$$\delta_1 d_1^2 = \varepsilon_0 g_1^2 + \varepsilon_1 h_1^2 + \varepsilon_2 l_1^2, \qquad (2.13)$$

where $\delta_1 = \frac{<\alpha,\alpha>}{|<\alpha,\alpha>|} = \pm 1$. Taking derivative with respect to s in (2.13) and using (2.11) and (2.12), we easily get

$$h_1 = \frac{\varepsilon_1}{\tau}\left(\varepsilon_2 - \delta_1\left({d_1'}^2 + d_1 d_1''\right)\right). \qquad (2.14)$$

We give the following corollary of Theorem 2.4:

Corollary 2.2 Let γ be a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$ in Minkowski $3-$space R_1^3 and α be its conjugate mate. Then,

i) If α is a hyperquadrical curve, then

$$h_1 = \frac{\varepsilon_1 \varepsilon_2}{\tau} \tag{2.15}$$

ii) If α is a rectifying curve, then $h_1 = 0$.

Proof. We assume that γ is a unit speed curve with Frenet apparatus $\{\kappa > 0, \tau \neq 0, T, N, B\}$ and α is the conjugate mate of γ.

i) If α is a hyperquadrical curve, then $d_1 = constant$. So from (2.14), we easly get (2.15).

ii) If α is a rectifying curve, then $\delta_1 d_1^2 = \varepsilon_2 s^2 + c_1 s + c_2$ for some constants c_1 and $c_2 \neq 0$. From (2.14), we obtain $h_1 = 0$.

3. Conclussion.

As is well known, position vectors find applications throughout mathematics, engineering, and the natural sciences. With this motivation, we study the position vectors of the natural and conjugate mates of a non-null curve in Minkowski 3-space. We characterize a non-degenerate helix, non-null Bertrand curves, hyperquadrical curves, and rectifying curves by means of the position vectors of the natural and conjugate mates of a non-null space curve.

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CHAPTER 2

HYPERELASTIC CURVES IN GALILEAN 3-SPACE

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Hyperelastic curves are defined as curves obtained by solving a more general form of the variational problem, where the integral of the square of curvature, under suitable initial conditions, is treated as a critical point. In the differential geometry literature, these curves are examined using various metrics tailored to the structure of the space and analyzed based on their geometric properties in both Euclidean and non-Euclidean contexts. Investigating hyperelastic curves in Galilean 3-space under the Galilean metric provides a unique perspective to thoroughly understand the behavior of these curves within distinct metric frameworks and to explore the evolution of their geometric properties. Accordingly, we study the curvature energy action in Galilean 3-space, identify hyperelastic curves as critical points of this action, and analyze their geometric behavior under this specific metric structure. The characterization of hyperelastic curves is achieved through an Euler-Lagrange equation in Galilean 3-space, with examples provided to illustrate their theoretical and geometric features.

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1. Introduction

The concept of curvature energy action refers to a functional that measures the energy associated with the curvature of a curve. It is defined as the integral of a function of its curvature along the entire length of the curve. This energy action is fundamental in variational problems where the aim is to find curves that minimize or maximize this energy, and leads in particular to the study of elastic and hyperelastic curves, which are the subject of this chapter. By characterizing curves by means of their curvature energy, it is possible to discover the equilibrium forms and stability properties of various geometric and physical systems, making it an important tool in the study of differential geometry and elastic phenomena.

We consider a space curve β with curvature κ and torsion $\tau.$ Then the general functional represented by the curvature and torsion of the curve β, $\int f(\kappa, \tau, \kappa', \tau', \ldots)ds,$ is known a Hamiltonian for β (Capovilla et al., 2002; Tükel, 2019). This Hamiltonian formulation allows us to describe the behavior of the space curve β through a variational approach, where the energy of the system is determined by the curvature κ, torsion τ, and their higher-order derivatives. By analyzing this functional, we can derive the Euler-Lagrange equations that govern the equilibrium states of the curve, offering insight into the geometric and physical properties of β. Such an approach is fundamental in studying the stability and behavior of elastic curves and has applications in fields like mechanics and differential geometry. One of the significant applications of these Hamiltonians is the functional $\int (\kappa^r + \lambda)ds$, for the natural number $r \geq 2$ and the Lagrange multiplier λ, whose critical points under sutable boundary conditions are known as hyperelastic (or r-elastic) curves. In the case of $\lambda = 0$, solutions of the functional are known as free hyperelastic curves. Also, the solutions of the functional are classical elastic curves when $r = 2$ (Barros et al., 1998; Arroyo et al., 2003; Şahin et al., 2021). This field has been extensively studied by numerous researchers over the years and continues to be a vibrant area of ongoing research and innovation. For example, Arroyo et al., (2003) explore closed free hyperelastic curves within the hyperbolic plane, examining their properties and connections to Chen-Willmore rotational hypersurfaces. The authors employ variational methods to study the geometric evolution of these curves, highlighting their mathematical and physical significance in the context of hyperbolic geometry. Arroyo et al., (2003) investigate closed generalized elastic curves in the unit 2-sphere, deriving their characterization through variational principles and analyzing their geometric properties within the framework of differential geometry. The paper written by Ferrández et al. in (2006) investigates the motion of relativistic particles described by an action that depends on the curvature and torsion of their paths in three-dimensional pseudo-Riemannian space forms. The authors derive Euler-Lagrange equations for these systems and discuss the moduli space of solutions, employing Killing vector fields and variational principles. Özkan Tükel et al. (2020) investigate a constrained variational problem in the context of a 2-dimensional null cone within Minkowski 3-space, deriving and solving the Euler-Lagrange equation for hyperelastic curves and exploring the application of Killing vector fields to describe these curves' geometric properties. Later, Kağızman and Yücesan (2022) generalized the study to a 3-dimensional lightlike cone.

Galilean geometry, inspired by Galileo Galilei, is a form of non-Euclidean geometry frequently applied in classical mechanics. Unlike Euclidean geometry, it treats time and space in a distinct way, laying the groundwork for the development of modern physics concepts. Studies on bending energy functionals are frequently conducted in Galilean 3-space, where the geometric properties of curves and energy minimization problems are examined in detail within this context (see, Şahin, 2013; Gürbüz, 2013;

Tükel and Turhan, 2020; Çivi et al., 2021, etc.). Tükel and Turhan (2020) study classical Euler-Bernoulli elastic curves within the context of Galilean 3-space, deriving the Euler-Lagrange equations for bending energy and solving them to illustrate examples of elastic curves in this non-Euclidean setting. Tükel and Turhan (2023) investigate the geometry of natural Hamiltonians for curves in Galilean and Pseudo-Galilean 3-space, deriving Euler-Lagrange equations to characterize elastic curves and providing detailed solutions through variational principles.

In the present work, we investigate the extremals of the functional $\int (\kappa^r + \lambda)ds$ for curves under certain boundary conditions in Galilean 3-space. This study is motivated by the desire to understand the geometric behavior of curves that arise as critical points of this variational problem under the Galilean metric structure. By employing the principles of the calculus of variations, we derive the associated Euler-Lagrange equations, which serve to characterize the solutions of this functional. These solutions are then defined as Galilean hyperelastic curves, a generalization that offers insights into their intrinsic geometric properties. Furthermore, we provide illustrative examples to demonstrate the application of our theoretical framework, showcasing the relevance and implications of Galilean hyperelastic curves within the context of differential geometry.

2. Conceptual Framework

Galilean geometry, rooted in the principles established by Galileo Galilei, stands as a non-Euclidean framework widely applied in classical mechanics. Unlike the fixed notions of distance and angle consistency in Euclidean geometry, Galilean geometry introduces a distinct treatment of spatial and temporal dimensions. This alternative perspective on space and time serves as a foundational element that bridges the gap to more complex geometric and physical theories. Galilean space is a specific example of a real Cayley-Klein space, characterized by unique geometric properties that differentiate it from classical Euclidean space. When traditional geometric findings from Euclidean space are adapted to Galilean settings, they reveal profound and impactful insights.

Let G_3 be the Galilean 3-space and $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ vectors in G_3. The Galilean scalar product of u, v is written as

$$< u, v >_{G_3} = \begin{cases} u_1 v_1 & if \quad u_1 \neq 0 \; or \; v_1 \neq 0, \\ u_2 v_2 + u_3 v_3 & if \quad u_1 = 0 \; and \; v_1 = 0. \end{cases}$$

For simplicity and a cleaner appearance, subscript notation will be omitted from this point forward. The vectors u and v are said to be perpendicular in the Galilean sense if $< u, v >= 0$. The vector $u = (u_1, u_2, u_3)$ is called as isotropic (non-isotropic) if $u_1 = 0$ $(u_1 \neq 0)$. Any unit non-isotropic vector has the form $u = (1, u_2, u_3)$. The Galilean norm of the vector u is calculated as follows

$$\|u\| = \begin{cases} |u_1| & if \quad u_1 \neq 0, \\ \sqrt{u_2^2 + u_3^2} & if \quad u_1 = 0, \end{cases}$$

(Keleş, 2004; Yaglom, 1979).

A curve $\beta: I \subset R \rightarrow G_3$ is called as admissible if it has no inflection points and no isotropic tangents (Erjavec, 2014). For a unit speed admissible curve $\beta(s)$ parametrized by

$$\beta(s) = \big(s, \beta_2(s), \beta_3(s)\big),$$

where s is the arclength parameter of β**,** we have the curvature $\kappa(s)$ and the torsion $\tau(s)$ as follows

$$\kappa(s) = \sqrt{\beta_2''(s) + \beta_3''(s)}$$

and

$$\tau(s) = \frac{\det(\beta'(s), \beta''(s), \beta'''(s))}{\kappa^2(s)}.$$

The Galilean Frenet frame of the curve $\beta(s)$ is given by

$$T(s) = \beta'(s),$$
$$N(s) = \frac{1}{\kappa(s)}\beta''(s),$$
$$B(s) = \frac{1}{\kappa(s)}\left(0, -\beta_3''(s), \beta_2''(s)\right),$$

where *T, N*, and *B* are respectively known as the tangent vector, principal normal vector and binormal vectors of $\beta(s)$. The Frenet equations of β(s) are given in matrix form as

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{1}$$

(Tükel and Turhan, 2020).

3. Results

This section is devoted to deriving the Euler-Lagrange equations that characterize the critical points of the curvature energy action in Galilean 3-space $\boldsymbol{G_3}$. These equations serve as the foundation for understanding Galilean hyperelastic curves, which are defined as extremals of this functional. The study of such curves provides insight into how curvature-driven energy optimization operates under the Galilean metric, revealing their unique geometric properties and behaviors. Through this derivation, we aim to establish a deeper connection between the variational principles and the geometry of curves in non-Euclidean metric spaces.

Now, we consider admissible curves in G_3 defined on the fixed interval $[0, L]$. A hyperelastic curve β with curvature κ and speed ν is a critical point of the following bending energy functional

$$\mathcal{F}(\beta) = \int_{\beta} \kappa^{\mathrm{r}}(s)ds = \int_{0}^{L} \kappa^{\mathrm{r}}(t)\nu dt$$

with fixed length and boundary conditions. We consider Ω denotes the set of the admissible curves

$$\beta : [0, \mathrm{L}] \subset R \to G_3$$

with $\beta(i\mathrm{L}) = p_i$, $\beta'(i\mathrm{L}) = v_i$ for $p_i \in G_3$ and $v_i \in T_{p_i}G_3$, $i = 0, 1$, and Ω_u the subspace of unit speed curves. Then $\mathcal{F}^{\lambda}: \Omega \longrightarrow R$

$$F^{\lambda}(\beta) = \frac{1}{2}\int_{\beta} \left(\|\beta''(t)\|^{\mathrm{r}} + \Lambda(t)(\|\beta'(t)\|^2 - 1)\right)dt.$$

One version of the Lagrange multiplier principle says a minimum of $\mathcal{F}$ on Ω_u is a critical point for $\mathcal{F}^{\lambda}(\beta)$ for some $\Lambda(\mathrm{t})$ which depends on constant λ (Singer, 2008). Note that a natural Hamiltonian produced by the scalar product $< T', T' >$, where T is the tangent vector of a curve $\beta \in \Omega \subset G_3$ is defined as the problem of finding critical points of the total squared curvature functional $\int_{\beta} \kappa^2 ds$ acting on Ω and the solutions to this variational problem are elastic curves in G_3. This also corresponds to the case $\mathrm{r} = 2$ of the $\mathcal{F}$ functional. Therefore, the functional $\mathcal{F}$ can also be considered as a generalization of the natural Hamiltonian functional. Tükel and Turhan (2020) reorganized the bending energy functional using the Lagrange multiplier principle with a Lagrange constant Λ dependent on length and minimized it by using Singer's approach. They derived Euler Lagrange equation that characterizes elastic curves in G_3 and they solve the equations. According to the characterization for an elastic curve β in G_3 is given by the Euler Lagrange equation

$$\kappa'' - \frac{c^2}{\kappa^3} - \frac{\lambda}{2}\kappa = 0, \tag{2}$$

where $\lambda = 2\Lambda$ and c are constants.

To find the solutions of the variational problem, the first variation must be computed. For this purpose, we consider a map $\beta: (-\varepsilon, \varepsilon) \times I \rightarrow G_3$, so that $(w,t) \rightarrow \beta\,(w,t) = \beta_w\,(t)$ and the curve $\beta_w\,(t)$ goes throughout β provided that $\beta(0,t) = \beta(t)$. Then $\beta_w\,(t)$ is a variation of β. The vector fields

$$V(w,t) = \frac{\partial\beta(w,t)}{\partial t} \text{ and } W\,(w,t) = \frac{\partial\beta(w,t)}{\partial w}$$

can be defined, where $V(0,t) = \beta'(t)$ and $W(t) = W(0,t)$ is a variational vector field along $\beta(t)$ so that $\left.\frac{\partial\beta(w,t)}{\partial w}\right|_{w=0} = W\,(t)$ (Arroyo et al., 2003; Turhan and Tükel, 2023). If s denotes the arclength parameter, then $\beta(w,s)$, $\kappa^2(w,s)$, $V\,(w,s)$, etc. can be written for the corresponding reparametrizations, where $s \in [0,L]$ and L is arc length of β.

Then we have the following theorem.

Theorem 1. A hyperelastic curve with curvature $\boldsymbol{\kappa}$ in the Galilean space $\boldsymbol{G_3}$ is characterized by the Euler Lagrange equation for $r \geq 2$

$$\frac{r(r-1)}{2}\kappa^{r-2}\kappa'' + \frac{r(r-2)(r-1)}{2}\kappa^{r-3}(\kappa')^2 - \frac{r}{2}\kappa^{3-3r}c^2 - \frac{\lambda}{2}\kappa = 0. \tag{3}$$

Proof. We assume that β is an extremum of $\mathcal{F}^\lambda$ and W is an infinitesimal variation of β. Then we have

$$\partial\mathcal{F}^\lambda(W) = \frac{\partial}{\partial\varepsilon}\mathcal{F}^\lambda(\beta + \varepsilon W)\Big|_{\varepsilon=0} = 0.$$

Thus we obtain

$$0=\frac{1}{2}\frac{\partial}{\partial\varepsilon}\int_{\beta}\big(\|(\beta+\varepsilon W)''\|^{\mathrm{r}}+\Lambda(\|(\beta+\varepsilon W)'\|^{2}-1)\big)dt.$$

The steps of the derivation proceed as follows through the variation of the curvature energy functional under the Galilean metric:

$$0=\int_{\beta}\frac{r}{2}<\beta'',\beta''>^{\frac{r-2}{2}}<\beta'',W''>+\Lambda<\beta',W'>dt.$$

If partial integration is applied after taking the variation, we obtain

$$0=\int_{0}^{\ell}<E[\beta],W>ds+(S[\beta,W])|_{0}^{L}$$

where

$$E[\beta]=\left(\left(\tfrac{r}{2}\kappa^{r-2}\beta''\right)'-\Lambda\beta'\right)' \tag{4}$$

and

$$(S[\beta,W])|_{0}^{L}=\frac{r}{2}<\kappa^{r-2}\beta'',W'>\Big|_{0}^{L}+<-(\kappa^{r-2}\beta'')'+\Lambda\beta',W>|_{0}^{L}.$$

Taking into consideration (1), we get the following derivatives of β;

$$\beta''=\kappa N,$$

$$\beta'''=\kappa'N+\kappa\tau B$$

and

$$\beta''''=(\kappa''-\kappa\tau^{2})N+(2\kappa'\tau+\kappa\tau')B.$$

Substituting these derivatives into (4), we arrive at

$$E[\beta] = -\Lambda' T + \left(\frac{r(r-1)}{2}\kappa^{r-2}\kappa'' + \frac{r(r-2)(r-1)}{2}\kappa^{r-3}(\kappa')^2 - \frac{r}{2}\kappa^{r-1}\tau^2 - \Lambda\kappa\right)N + \left(r(r-1)\kappa^{r-2}\kappa'\tau + \frac{r}{2}\kappa^{r-1}\tau'\right)B = 0$$

Using the fact that the quantity $E[\beta]$ vanishes identically when β is a critical point of the functional $\mathcal{F}^{\lambda}$ (Griffiths, 2013) and T, N and B are linear independent, we have

$$-\Lambda' T = 0, \tag{5}$$

$$\frac{r(r-1)}{2}\kappa^{r-2}\kappa'' + \frac{r(r-2)(r-1)}{2}\kappa^{r-3}(\kappa')^2 - \frac{r}{2}\kappa^{r-1}\tau^2 - \Lambda\kappa = 0 \tag{6}$$

and

$$r(r-1)\kappa^{r-2}\kappa'\tau + \frac{r}{2}\kappa^{r-1}\tau' = 0. \tag{7}$$

From (5), we obtain

$$\Lambda = \frac{\lambda}{2}, \tag{8}$$

where λ is a arbitrary constant. Multiplying by $\frac{2}{r}\kappa^{r-1}$ both of sides of (5)

$$2(r-1)\kappa^{2r-3}\kappa'\tau + \kappa^{2r-2}\tau' = 0.$$

Thus, we arrive at

$$\left(\kappa^{2(r-1)}\tau\right)' = 0,$$

that is,

$$\kappa^{2(r-1)}\tau = c, c = const. \tag{9}$$

Combining (6), (8) and (9), we derive the Euler Lagrange equation (3).

One can see that Euler Lagrange equation (3) corresponds to (2) when $r = 2$.

Example 1. A straight line β in G_3 (see, Şahin and Dirişen, 2017) is a critical point of the functional $\mathcal{F}$, that is, β is a hyperelastic curve.

Example 2. Let β be a planar curve in G_3 (see, Şahin and Dirişen, 2017). If β is is a critical point of the functional $\mathcal{F}$, then, β satisfies the following Euer Lagrange equation

$$\frac{r(r-1)}{2}\kappa^{r-2}\kappa'' + \frac{r(r-2)(r-1)}{2}\kappa^{r-3}(\kappa')^2 - \frac{\lambda}{2}\kappa = 0.$$

Example 3. Any circular helix or W-curve in G_3 (see, Şahin and Dirişen, 2017) is a solution of the functional $\mathcal{F}$. Thus, any circular helix or W-curve is a hyperelastic curve with $\lambda = -r\kappa^{2-3r}c^2$.

Example 4. Let $\beta(s) = \left(s, \frac{s - sinscoss}{4}, \frac{sin^2 s - s^2}{4}\right)$ be a curve in G_3. The curvature and torsion of $\beta(s)$ is $\kappa(s) = sins,\ \tau(s) = 1$ (see, Şahin and Dirişen, 2017). This curve is a solution of the functional $\mathcal{F}$ for $s = \frac{\pi}{2} + k\,\pi, k \in Z$ with $\lambda = -r$. For this parameter, β determines a hyperelastic curve.

Conclusion

In this study, the extremals of the functional $\int (\kappa^r + \lambda)ds$ in Galilean space were examined under specific boundary conditions, and the solutions of this variational problem were characterized using Euler-Lagrange equations. The defined Galilean hyperelastic curves provided a detailed understanding of their geometric properties within the framework of the Galilean metric structure. The examples presented demonstrated the applicability of the theoretical findings and offered significant insights into the behavior of curves under different metric structures. These results not only lay the groundwork for further exploration of hyperelastic curves in

Galilean space but also offer a comparative perspective for studying such curves in other metric frameworks. We hope that these findings will contribute to new research directions in both mathematical physics and differential geometry.

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CHAPTER 3

OPTIMIZATION AND REGION DELINEATION BASED ON PROXIMITY TO POINTS: A STUDY ON THE PIZZA DELIVERY PROBLEM

Nuh ÖZBEY[4]

Narin ÇELİK[5]

Division of Space into Regions based on Proximity to Specified Points

Mathematics is a branch of science that deals with the logic of shape, quantity, and arrangement. Its influence can be seen in every aspect of our daily lives; from the functioning of our computers to the exterior architectural design of buildings, from the expression of art to the economic use of money, mathematics plays an important role in many areas (Erbaş, 2016). Perhaps the most important impact of mathematics on our daily lives is its contribution to problem-solving. Problem and problem-solving are an inevitable reality of life and are also considered a fundamental part of mathematics. A problem is actually a challenge with an unknown solution, arousing a desire in individuals to solve it and requiring time to solve (Dündar, 2020).

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An algorithm is a concept consisting of a series of sequential steps designed to produce a solution to a problem or achieve a predetermined result (Aytekin, 2018). A well-developed algorithm tailored to a specific problem allows reaching a logical solution by following the listed steps (Sara, 2019). Algorithms are widely used not only in the field of computers but also in many areas of daily life (Atlı, 2022). In real life, to perform a task, usually, plans are made in advance, and the stages of the task are determined, similar to following the steps specified in a recipe to prepare a favorite dessert, which resembles the use of an algorithm. In solving a problem encountered in the software field, algorithms determine how initial values will be obtained, how these values will be processed using which methods, and where the results obtained will be used (Akçay, 2019). It is important for an algorithm to be as short and understandable as possible (Dündar, 2020). When designing an algorithm, the first step is to examine the problem to be solved in detail and review all possibilities. Then, a simple and understandable solution path that will lead to the result with the minimum number of steps should be determined. Finally, the generated algorithm must be executable, and its ability to reach the correct result must be verified (Aytekin, 2018).

The relationship between algorithms and mathematical problem solving is important for the organized solution of mathematical problems. Mathematical problems often require following certain steps, and these steps are defined by algorithms. Algorithms divide the problem into smaller parts and guide the solution process by determining the order of these parts (Aydoğdu, 2020). For example, an algorithm used to solve an equation determines the steps based on the characteristics of the equation and the solution method, and progresses the solution process systematically. While algorithms make the solution of mathematical problems more effective and efficient, they also help reduce the complexity of problems (Taşçı, 2016). Therefore, algorithms play a critical role in the solution of mathematical problems (Ekim and Aydemir, 2016).

There are different views on what a problem is in research. For example, according to Altun (2020), problems can be related to both real life and mathematics. Real life refers to the world outside of mathematics. Different subject areas of new schools and universities also constitute the real life of daily life and our environment. According to Polya (as cited in Çimen, 2023), a problem is the conscious investigation of actions to reach the goal in the most suitable way. If a situation in the mind can be solved relatively easily with certain actions, it is not considered a problem. It can be said that there is a problem to be solved if it is not clearly known which studies will be conducted for the solution. According to Polya (as cited in Çimen, 2023), the problem-solving process consists of the following stages: Understanding the problem, Creating the solution plan, Implementing the plan, and Checking the accuracy of the solution (Çiftçi, 2016). In the Understanding the Problem stage, it is important to understand the given information and the desired outcomes. The individual may need to understand the problem in their own words and, if necessary, draw a shape or diagram. In the Determining the Method stage, it is necessary to select appropriate strategies and techniques to solve the problem. In the Planning stage, plans are made according to the determined method, and operations are carried out. In the Checking stage, the accuracy of the operations used is checked, and the result is compared with the previously estimated value (Altun, 2020).

The problem-solving process mentioned above can be applied in a wide range of areas, from mathematical problems to real-world scenarios. As an example, this research started with the pizza delivery problem, which is based on a scenario we encounter during Olympiad studies and can also be encountered in the real world. This problem is essentially related to a city map drawn on a surface organized in unit squares and three different pizza branches located on this map. The goal of the problem is to divide the city into regions between these pizza branches. However, this division should be done in such a way that any point on the given city map is assigned to the nearest branch for delivery. In solving this problem, it was also necessary to plan how an order from any point on the given city map would be directed to the nearest branch. In this context, determining and drawing the boundaries was also one of the fundamental issues of the project. It can be said that this project, which constitutes the beginning of the studies on the solution of the pizza delivery problem, has an important place in showing how mathematical problems can arise in daily life and how mathematical thinking skills can be applied to real-world problems.

Method

In this research, the method of mathematical modeling was used. The mathematical modeling method involves the examination of real-life events or realistic situations using mathematical methods. The process of mathematical modeling includes problem identification and definition, creation of the mathematical model, solution of the model, interpretation of the results, and evaluation of the model's accuracy (Sarı, 2021).

In this context, firstly, the real-life problem to be solved is clearly identified, and objectives are defined. Then, a model expressing the problem in mathematical terms is created. This model allows the analysis of the problem using mathematical equations, relationships, or graphs. Next, the created model is solved using mathematical methods, and the results are interpreted and associated with real-world applications. Finally, the accuracy of the model is examined, and steps are taken to improve or update the model if necessary. These steps ensure that mathematical modeling directs the process of solving complex problems in a systematic and effective manner (Erdoğan, 2018).

The methodology of this research, designed according to the mathematical modeling method, is presented in two stages. The first stage involves obtaining the generalization method to be applied when there are n branches on a plane. The second stage involves the development of an algorithm for implementing applications in one, two, and three-dimensional spaces for this problem. These stages address the complexity of mathematical modeling in different dimensions, covering the basic findings of the project.

Stage 1: Division of the plane into regions based on proximity among n branches

In this stage, data obtained from studies on the division based on proximity among n branches, while staying true to the original problem, are presented. The presentation here follows the logic of mathematical generalization, progressing from smaller to larger.

Division of the plane into 2 regions

When a plane is divided into two parts according to the algorithm specified in the method section, a similar division occurs as shown in Figure 1 below.

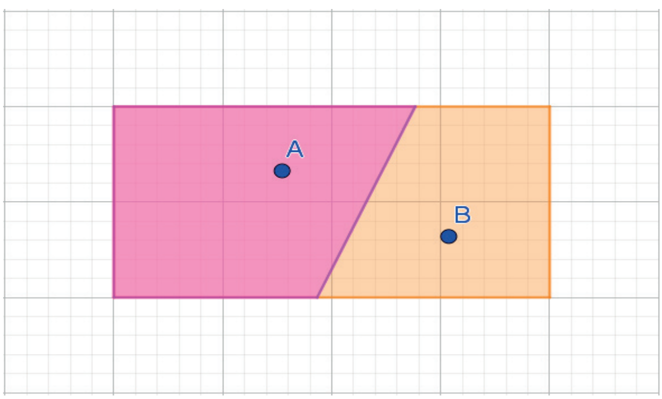

Figure 1. Division of the plane into 2 regions

In Figure 1, the region where points closer to point A than point B are located has been identified and shown in pink. Similarly, the region where points closer to point B than point A are located has been identified and shown in orange.

Division of the plane into 3 regions

In this case, drawings similar to the first step have been made, and the plane has been divided into 3 regions as shown in Figure 2 below.

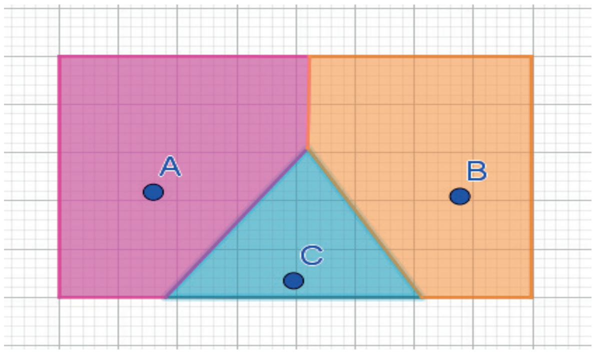

Figure 2. Division of the plane into 3 regions

In Figure 2, the region where points closer to point A than to other points are located has been identified and shown in pink. The same process has been applied to determine the regions of points B and C. According to this process, the region of point B is shown in orange, and the region of point C is shown in blue.

Division of the plane into 4 regions

In this case, drawings similar to the first step have been made, and the plane has been divided into 4 regions as shown in Figure 3 below.

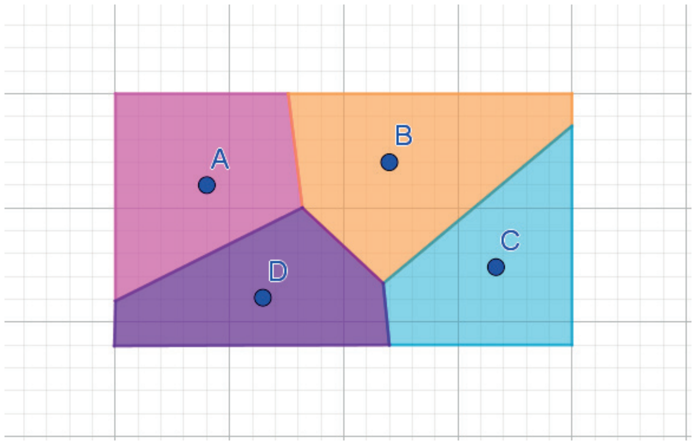

Figure 3. Division of the plane into 4 regions

The determination of the regions of the points in Figure 3 is similar to Step 1 above. According to this determination, the region of point A is shown in pink, the region of point B is shown in orange, the region of point C is shown in blue, and the region of point D is shown in purple.

Division of the plane into 5 regions

In this case, drawings similar to the first step have been made, and the plane has been divided into 5 regions as shown in Figure 4 below.

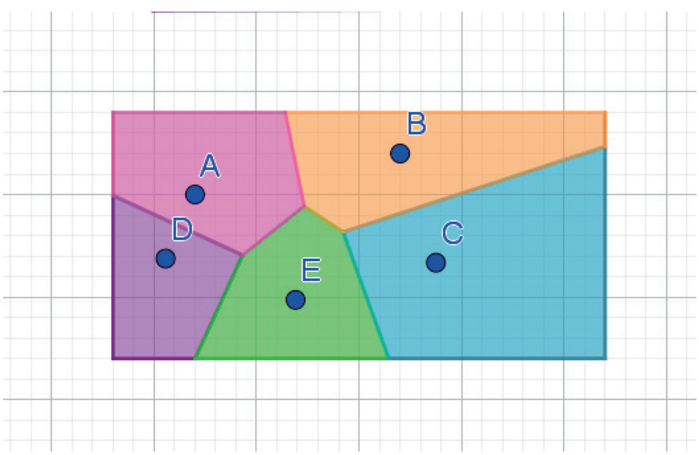

Figure 4. Division of the plane into 5 regions

The determination of the regions of the points in Figure 4 is similar to Step 1 above. According to this determination, the region of point A is shown in pink, the region of point B is shown in orange, the region of point C is shown in blue, the region of point D is shown in purple, and the region of point E is shown in green.

Stage 2: Formation of Regions in Different Dimensions

In this stage, the division of regions through lines in the plane is considered, similar to the division of regions through lines and three-dimensional spaces.

Determination of regions in one dimension

The division of a one-dimensional line segment into regions based on the closest distance to certain points occurs as in the following steps.

Step 1. In this step, a line segment AB as shown in Figure 5 has been drawn according to the algorithm specified in the method section.

Figure 5. Determination of the desired number (5) of points on the line segment

Points C, D, E, F, and G on the line segment AB in Figure 5 have been determined.

Step 2. In this step, the process of determining the midpoint of the points on the line segment AB, as shown in Figure 6, is performed. Below, the midpoint of each point on the line segment AB with its neighboring points is found, and the green-colored points H, I, J, K, L, and M are created.

Figure 6. Determination of the midpoint of the identified points

In Figure 6, the midpoint of each point on the line segment AB with its neighboring points is found, and the green-colored points H, I, J, K, L, and M are created.

Step 3. In this step, as shown in Figure 7, the selected line segments are colored with different colors so that each point remains in the center of the region closest to it.

Figure 7. Division of the line segment according to the proximity of the selected points

In Figure 7, the regions near points A are shown in blue, the regions near point C are shown in yellow, the regions near point D are shown in red, the regions near point E are shown in purple, the regions near point F are shown in orange, the regions near point G are shown in light blue, and the regions near point B are shown in brown.

Determination of regions in three dimensions

The division of 3-dimensional space into regions based on the proximity of certain points was carried out in accordance with the developed algorithm, as shown in the following steps.

Step 1. In this step, a cube representing space, as shown in Figure 8, was drawn, and the different number of points inside this cube were colored with the same color as the points closest to them.

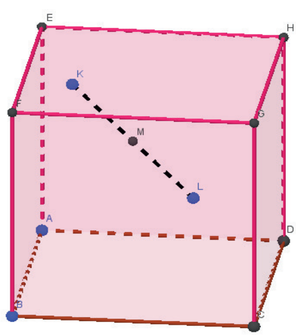

Figure 8. The midpoint created inside the cube

In Figure 8, the line segment between points K and L, created with the GeoGebra program, is drawn, and the midpoint M is determined.

Step 2. Then, a plane passing through point M and perpendicular to line segment KL is drawn, as shown in Figure 9.

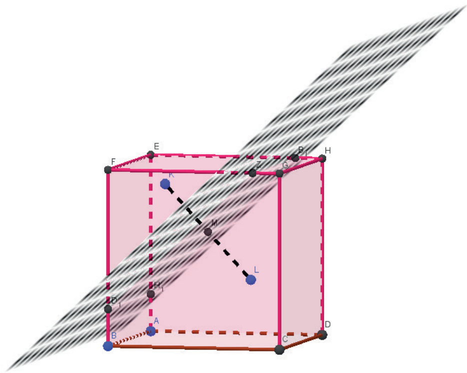

Figure 9. Cube and plane passing through the center point

In Figure 9, a plane passing through point M and the line segment KL is drawn, and the points where this plane intersects with the cube are determined.

Step 3. Then, in Figure 10, a quadrilateral representing the cross-section is drawn using the points where the plane intersects with the cube.

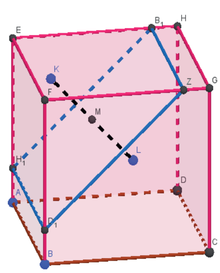

Figure 10. Intermediate section dividing the cube

In Figure 10, the parallelogram BDZH represents the cross-section of the cube with the plane. This parallelogram shows the boundary in the process of dividing the cube into regions based on the closest point criterion.

Step 4. Finally, the regions close to points K and L are separated, creating the two spatial parts shown in Figure 11 below.

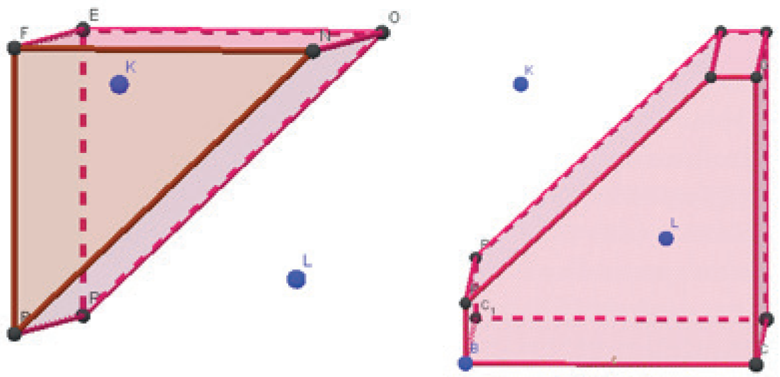

Figure 11. The cube divided into two regions

In Figure 11, the cube representing the space is divided into two parts, with points K and L marking the division.

Findinngs

The findings of this study, designed in accordance with the mathematical modeling method, consist of three main stages. The first stage involves geometrically presenting the problem with three pizza branches, the second stage involves obtaining the operations to be carried out when there are n branches on the plane, and the third stage involves creating an algorithm for the applications of the problem in one, two, and three-dimensional spaces.

Constructing the Mathematical Model of the Problem

In this stage, the process of creating the necessary geometric drawings for solving the problem using the GeoGebra program is explained step by step. This stage, consisting of 4 steps, involves creating drawings and obtaining data according to the following sequence.

a. Constructing the Mathematical Model of the Problem

In this step, the concepts in the original problem are converted into mathematical terms, and a mathematical model representing the problem is drawn. This model, drawn with the help of GeoGebra, is shown in Figure 12 below.

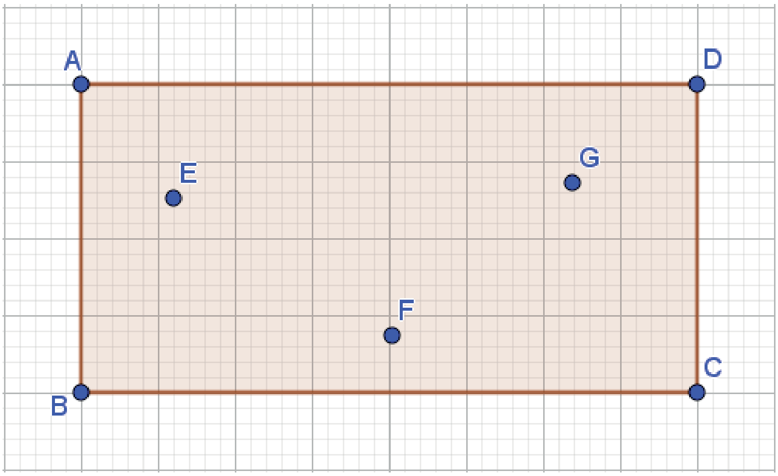

Figure 12. The mathematical model of the problem

Figure 12 shows the original problem. Here, the task is to divide the rectangle ABCD, representing the city, into 3 regions based on their proximity to the points E, F, and G, representing the pizza branches.

b. Drawing lines and determining midpoints

In this step, to solve the problem, each point was connected with the other points using a line segment, and the midpoints of these line segments were determined as shown in Figure 13.

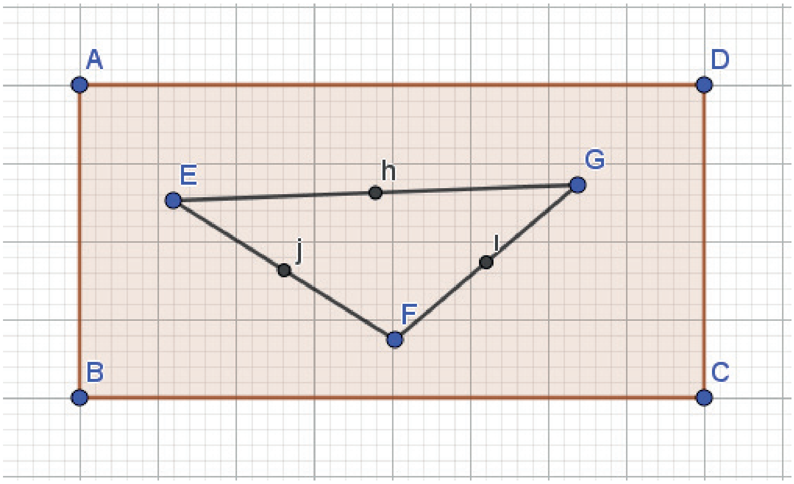

Figure 13. Determination of Midpoints between Points

Figure 13 shows the second step of the drawings made for solving the problem. Here, each of the points E, G, and F within the ABCD rectangle is connected with the other points excluding itself, and using appropriate menus in the GeoGebra program (such as the midpoint finding command), midpoints of each line segment are determined.

c. Drawing the Boundaries Using Midpoints

In this step, perpendicular lines are drawn to the line segments drawn in the previous step and their midpoints, as shown in Figure 14.

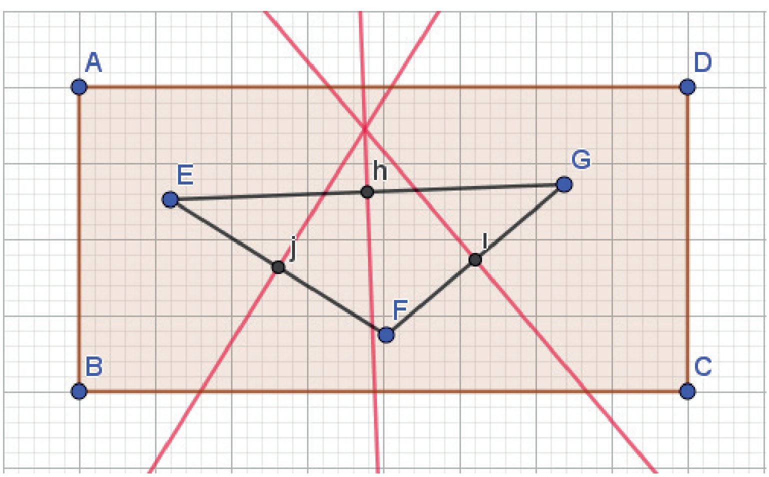

Figure 14. Drawing the Initial Boundaries Using Midpoints

Figure 14 shows the perpendicular lines drawn to the midpoints we previously determined, with the lines displayed in red.

d. Determination of Regions

In this step, the boundary regions of each point were determined based on the drawn perpendicular lines. The region of point E is represented in blue, the region of point F in green, and the region of point G in orange, as modeled in Figure 15.

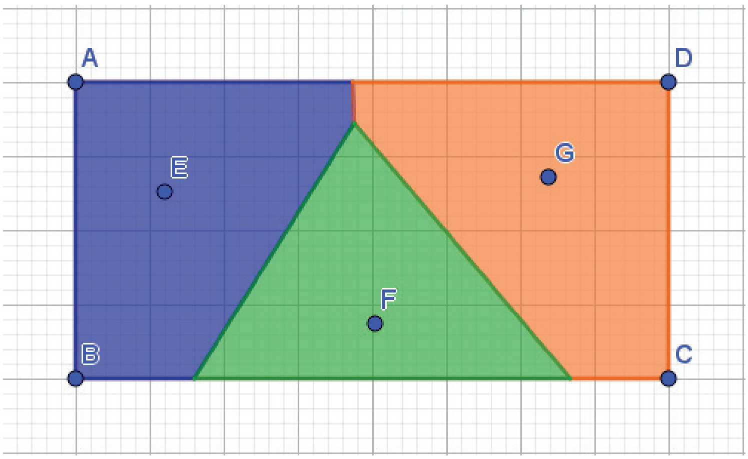

Figure 15. Determination of Regions

In Figure 15, the boundaries between points E, F, and G were determined using the perpendicular lines drawn in the previous step, and the regions formed by each point are modeled with polygons in different colors.

The Different Dimensional Situation of the Problem

In this stage, studies were conducted on how the operations performed for n points in a two-dimensional plane can be carried out in different dimensions, and a general model was created regarding these operations. This model provides a guide on how operations can be performed in different dimensions, offering a more comprehensive approach that can be used in solving mathematical and geometric problems.

Example 1: When there are three points with known positions in a three-dimensional space, the division into regions based on proximity to points will be as shown in Figure 16 below.

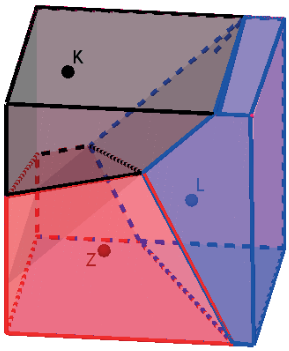

Figure 16. Division of the cube into three separate regions

In Figure 16, the cube representing the space is divided into regions, with the region of point K shown in red, the region of point L shown in blue, and the region of point Z shown in green.

Example 2: When there are 4 points in a 3-dimensional space, the division of regions based on proximity to points will be as shown in Figure 17.

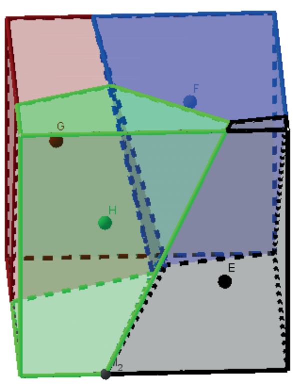

Figure 17 shows the cube divided into four separate regions.

The regions are indicated by the colors of the points within the cube: the region of point E is shown in black, the region of point F is shown in blue, the region of point H is shown in green, and the region of point G is shown in red.

Results and Discussion

The research conducted in this study, based on the pizza shop problem, has led to the development of effective algorithms for determining points in a space with a certain dimension that are closer to other points. Additionally, by focusing on a problem given in two dimensions, the study has examined the proximity of any point in different dimensions to the specified points. The analyses indicate that the distance to the specified points can be geometrically determined, and the partitioning process can be effectively performed using the developed algorithm. This situation is similar to the study conducted by Taşçı (2016), which examined the classification performance of nearest neighbor algorithm parameters using computer programs. The study emphasizes that the geometric partitioning performed is understandable and has a concrete form.

Another result of this study is the identification of regions in up to three-dimensional space where points are closer to a specified point than to other predetermined points. These findings indicate that as the dimensionality of the space increases, the computational complexity also increases, but effective algorithms can manage this complexity. In the study, a problem encountered in daily life was solved by creating a mathematical model and generalized to different dimensions. Mathematical models simplify complex systems, making them easier to work with and optimizing the solution process. This feature is also emphasized by Sarı (2021), who, in his study discussing people's views on mathematical modeling, states that those who prioritize the modeling process can solve complex situations more easily through modeling.

Based on the results of the research conducted using the pizzeria problem as a basis, the following recommendations can be made: Further work can be done to extend the algorithms used in two-dimensional problems to other dimensions. Especially in three-dimensional space, focusing on the complexity and efficiency of algorithms would be beneficial. This can help create a general solution strategy and effectively solve problems in larger dimensions. The research results provide a basis for developing new mathematical and algorithmic approaches that can be used in solving similar problems. In this regard, it is recommended to explore alternative methods for examining the proximity of points and to improve existing methods. In future research, focus can be given not only to geometric problems like the pizzeria problem but also on how such algorithms and methods can be applied in other areas. For example, investigating how these approaches can be applied in different disciplines such as data analysis, image processing, or robotics could be beneficial.

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CHAPTER 4

ALGORITHM DEVELOPMENT STUDY USING MATHEMATICAL GENERALIZATION METHOD (EXAMPLE: SUM OF UNIT FRACTIONS)

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Elifsu TOZLUKLU[7]

Algorithm Design Using Mathematical Generalization Method (Example: Sum of Unit Fractions)

Unit fractions are a fundamental concept in mathematics with a significant historical development. Since ancient times, many thoughts and studies have been conducted on the basic rules and calculations related to unit fractions. In this context, it is known that in ancient Egypt in the 3rd millennium BC, unit fractions were used to solve mathematical problems related to rational numbers and fractions. During those times, the use of fractions was widespread, especially for solving practical needs such as construction and trade (Seyhan, 2021). The use of fractions in daily life by the people of Egypt has formed a fundamental tool for practical mathematical calculations. Mathematical calculations and the use of fractions in ancient Egypt developed, especially

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to meet practical needs in daily life (Danacı, 2021). Fractions have played an important role in fundamental activities such as agriculture, construction, and trade in Antic Egyptian society. Especially, fractions were frequently used in issues such as the flooding of the Nile River irrigating agricultural lands and the sharing of water. Mathematical problems such as measuring land and dividing water were solved with the mathematical expressions of fractions (Seyhan, 2021). Moreover, the ancient Egyptians also carried out large construction projects such as pyramids. It can be thought that fractions were used in these types of projects for material calculations, planning construction processes, and similar tasks. Similarly, during trade at that time, fractions were commonly used in issues such as the distribution of products and pricing, and fractions played an important role in solving many problems encountered in daily life. Ancient Greek mathematicians also studied fractions like the ancient Egyptians and developed many theorems on them. In particular, scientists such as Euclid and Archimedes made significant contributions to mathematics regarding fractions. Especially Archimedes' work on fractions played an important role in developing methods for finding their approximate values (Feyzioğlu, 2019).

At the end of antiquity and especially in the Middle Ages, Muslim mathematicians also conducted intensive studies on fractions, and these studies played an important role in the history of mathematics. Especially during the Golden Age of Islam, between the 8th and 14th centuries, various research was conducted on the theoretical and practical applications of unit fractions (Yıldız, 2021). Mathematicians of this period developed various methods for the generalization of fractions, performing the four operations among fractions, calculating the approximate values of fractions, and solving algebraic problems. Prominent figures such as Al-Khwarizmi, Al-Battani, and Al-Biruni increased the knowledge in this field by writing works on fractions. The work of Muslim mathematicians has made significant contributions to Western mathematics in the Middle Ages and beyond, contributing to a deeper understanding and use of fractions (Göl, 2022).

Studies on fractions continued during the Middle Ages and the Renaissance periods. In medieval Europe, fractions were widely used for solving mathematical problems and commercial calculations (Öndin, 2021). However, some complex problems related to fractions emerged during the Middle Ages, especially difficulties related to proportions and addition and subtraction of fractions. In the Renaissance period, new approaches were developed regarding the representation of fractions in decimal form. During this period, various methods emerged for representing and calculating fractions in decimal format. This allowed mathematical calculations to be performed more easily and accurately (Okuyucu, 2021). Throughout history, the development in mathematical thought and methods has not been limited to fractions alone. There have been significant changes in the general understanding of mathematics. During the Renaissance period, mathematics adopted a more systematic approach, and significant progress was made not only in fractions but also in other areas such as algebra, geometry, and probability. Mathematicians of this period developed new methods for the decimal representation and operations of fractions while also approaching mathematical thought from a broader perspective. Especially, representing fractions in decimal format to facilitate their practical use has increased the accuracy of mathematical calculations in fields such as trade, engineering, and science, and accelerated the solution of daily life problems (Öndin, 2021). These advancements during this period further solidified the use of fractions, which are one of the cornerstones of modern mathematics. Even today, the use of fractions in various fields of mathematics and practical life carries traces of these developments that began in the Renaissance period. The development of unit fractions in the history of mathematics has laid the foundation for the use of fractions in modern mathematics. Today, fractions are widely used in solving mathematical problems and in various areas of daily life.

Unit fractions, which form the basis of fractions, are simple fractions with a numerator of 1 and hold a more special position compared to other types of fractions. This characteristic becomes more pronounced with their fixed numerators and their immutability. In this section, considering the mentioned qualities of unit fractions, unit fractions have been selected as the focus of the study. This choice has laid the foundation for the development of methods to obtain new unit fractions from unit fractions and for a deeper investigation into their relationship with the addition operation. This approach will lead to a better understanding of the mathematical properties of unit fractions and contribute to increasing the knowledge that will lead to new discoveries in this field. Unit fractions not only underpin mathematical thinking but also open the door to understanding more complex fraction structures. Therefore, research on unit fractions offers an important opportunity to deepen and expand mathematical thinking. Such research can play a key role in advancing mathematical theory and shedding light on new mathematical discoveries in the future. Thus, research on the properties and uses of unit fractions has the potential to contribute to the development of mathematical knowledge.

Method

In this study, the method employed is the mathematical generalization method, focusing on how unit fractions can be expressed as the sum of two unit fractions. Mathematical generalization involves expressing a specific mathematical knowledge or situation under a more general rule or model to make it applicable. This method allows for a broader understanding of a particular situation and the evaluation of its applicability in similar contexts. Consequently, mathematical generalization contributes to understanding relationships between specific cases and further exploring mathematical knowledge in depth. This process begins with the examination and analysis of specific examples, followed by the use of mathematical expressions to express the findings in a more general framework. The created mathematical model forms a structure that clearly expresses the characteristics and relationships of the examined situation. This model aims to generalize the results obtained from examples and make them applicable in a broader context. Finally, the created mathematical model is examined using various mathematical techniques to test its general validity under specific conditions. In this research, which is based on the method of mathematical generalization, the studies were conducted in four steps according to the method. The first step is the use of expanded fractions, the second is the examination of different notations, the third is the generalization of different notations, and the fourth is the proof of generalization and writing of the algorithm.

The Use of Expanded Fractions

In this stage, first, the properties of unit fractions and existing knowledge related to the topic were examined in detail. Then, the focus was on the properties identified to initiate the process of mathematical generalization. Initially, examples were worked on using the sampling method, and unit fractions were expanded with counting numbers from 1 to 20, focusing on the sums that give this form of the fraction. The results obtained were organized in tables, some of which are included in Table 1, to facilitate the review.

Table 1. Expanded Forms and Sums of 1/4 Fraction

Expanded Number	Expanded Fraction	Representation as Sum of Unit Fractions
1	1/4	Notation N/A
2	2/8	1/8 + 1/8
3	3/12	2/12 + 1/12 or 1/6 + 1/12
4	4/16	Notation N/A
5	5/20	4/20 + 1/20 or 1/5 + 1/20
6	6/24	4/24 + 2/24 or 1/6 + 1/12

In Table 1, the expanded form of the selected 1/4 fraction with the designated numbers and the ways in which these forms are obtained as the sum of 2 unit fractions are presented. Different unit fraction examples expanded with counting numbers up to 20 were developed separately in the research process. The first column of these tables contains the number at which the unit fraction is expanded, the second column contains its expanded form, and the third column contains the ways in which the designated unit fraction is obtained under desired conditions. These tables, used as a regular source of information gathering, have also contributed to the emergence of two problems encountered during the process. The first problem is the occurrence of the same sums representing the selected unit fraction multiple times. The second problem is the absence of a specific limit for the number to be used in the expansion process. Despite these problems, data obtained in this phase were used to obtain data suitable for generalization in the second phase.

Examination of Different Notational Practices

In the second stage of the research, using the data from the previous stage, examples were made to determine how many different ways unit fractions from 1/2 to 1/20 can be written as the sum of two unit fractions. For this purpose, the possible ways of expressing each unit fraction as the sum of two unit fractions were investigated, starting from 1/2 to 1/20. The results of this sampling were organized in a table format, as shown in Table 2 below.

Table 2. The number of times a unit fraction is obtained under the desired conditions

Desired Denominator	Number of Different Combinations	Desired Denominator	Number of Different Combinations
1	1	11	2
2	2	12	8
3	2	13	2
4	3	14	5
5	2	15	5
6	5	16	5
7	2	17	2
8	4	18	7
9	3	19	2
10	5	20	7

Table 2 contains data on the number of different ways unit fractions from 1/2 to 1/20 can be obtained as the sum of two unit fractions. After producing this table, sample examinations were conducted. In these examinations, it was observed that unit fractions with prime denominators could be written in different ways. For example, the unit fraction 1/5 can be written as the sum of unit fractions in 2 different ways: 1/10 + 1/10 and 1/6 + 1/30.

Generalization of Different Writing Numbers

In this stage, based on the data obtained in the previous step and observations made regarding non-prime numbers similar to prime numbers, it was decided to organize the data according to different writing numbers for unit fractions as the sum of two unit fractions. As a result of this organization, Table 3 below was obtained.

Table 3. Unit fractions, the denominators, and the number of different ways

Number of Different Combinations	Denominators
1	1
2	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47
3	4, 9, 25, 49
4	8, 27
5	6, 10, 14, 15, 16, 21, 22, 26, 33, 34, 35, 38, 39, 46
6	32
7	18,24
8	12, 28, 44, 45, 50

In Table 3, the right column shows the denominators of the unit fractions under investigation. The left column indicates the number of different ways these fractions can be written as the sum of two unit fractions. For example, the fraction 1/3 can be written in 2 different ways, while 1/4 can be written in 3 ways, and 1/27 can be written in 3 ways as well. Based on the data presented in this table, it has been determined that the number of different ways a unit fraction can be expressed as the sum of two unit fractions is related to the prime factors of the denominator of that fraction. This relationship will be detailed in the Findings section as a result of the study.

Proving the Generalization and Writing the Algorithm

After the examination and determination of the characteristics, efforts were made to prove the generalization regarding the number of ways a unit fraction can be written as the sum of two unit fractions. Algebraic expressions were used for this purpose, and instead of the previously tried fractions, the fractions 1/a and 1/b were considered as the sum of two unit fractions, and the correctness of the generalization was proven. This proof will be detailed in the findings section. After determining the number of ways of writing, studies were conducted on what these writings are, and an algorithm was developed to find the ways a unit fraction can be written as the sum of two unit fractions. This algorithm will be presented in detail in the findings section.

Findings

In this part of the study, the data obtained throughout the process will be presented. This presentation will be carried out in stages, following the order in the methodology section. The first stage will include the generalization regarding how many different ways a selected unit fraction can be written as the sum of two unit fractions, the proof of this generalization, and the algorithm developed to apply the general rule more quickly. In the second stage, a general rule regarding in which ways selected unit fractions can be written as the sum of two unit fractions will be presented, along with an algorithm to assist in applying this rule.

The number of ways a unit fraction can be written as the sum of two unit fractions

In this stage, a generalization will be presented based on the data obtained using the data tables mentioned in the methodology section.

Generalization: A unit fraction can be written as the sum of two unit fractions in a number of ways equal to half of one more than the positive divisors of the square of the number in the denominator. This can be expressed mathematically as;

The number of ways to express 1/k as the sum of two unit fractions can be found using the formula (C(k^2) + 1) / 2, where C(k) is the number of positive divisors of k.

Proof: Let a, b, and k be natural numbers such that the sum of the unit fractions 1/a and 1/b equals 1/k. We have:

1/a + 1/b = 1/k. Multiplying both sides by ab yields:

b/ab + a/ab = 1/k. Combining the fractions:

(a + b)/ab = 1/k, Which simplifies to:

(a + b) = ab/k. Rearranging terms gives:

ak + bk = ab. Isolating ak:

ak = ab - bk. Now, isolate b in terms of a and k:

ak = b(a - k). Dividing both sides by (a - k) gives:

b = ak/(a - k).

b = k(k^2)/(a - k). This expression can also be written as a sum:

b = k + k^2/(a - k).

Here, a and k are natural numbers. For b to be a natural number, k^2 must be divisible by (a-k). Therefore, the number of values that can be substituted for (a-k) should be equal to the number of factors of k^2. However, since a and b can be interchangeable, to avoid repeating the same sums, only half of the factors should be used.

Considering that the number of factors of a perfect square is always odd, the number of solutions should be half of one more than the number of factors of k^2.

Algorithm 1. It was proven in the previous section that a unit fraction can be written in different ways as the sum of two unit fractions, where the number of ways is half of the positive divisors of the number in the denominator plus one. The generalization obtained at this stage has been transformed into the algorithm presented in Figure 1 to provide a practical application.

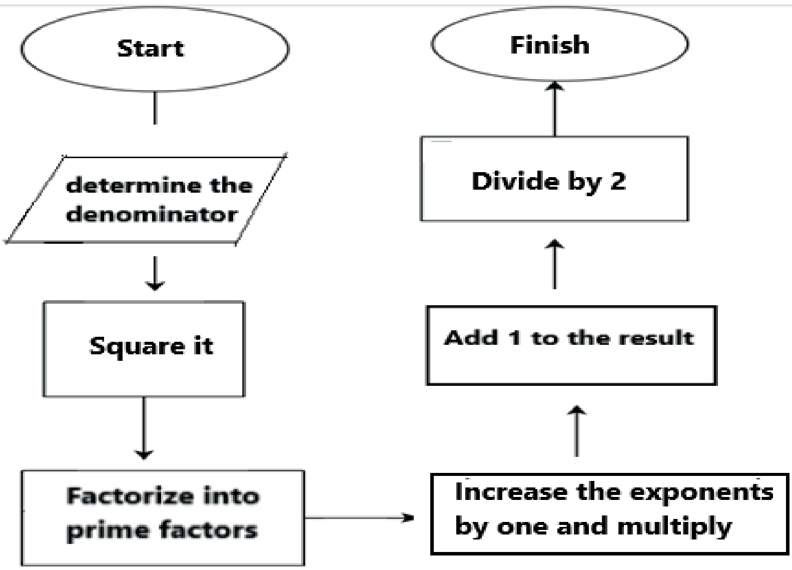

Figure 1. Flowchart of the algorithm used to find the total count

Figure 1 depicts the algorithm for finding the number of different ways a unit fraction can be written as the sum of two unit fractions. Following the development of this algorithm, a computer code in Python was generated, as shown in Figure 2, to quickly provide the result for larger numbers, aiming to facilitate usage.

```python
sayi = int(input("sayı giriniz"))

liste = []
liste1 = []
kare  = sayi*sayi
for i in range(1, kare+1):
    if kare % i == 0:
        liste.append(i)
        liste1.append(i+sayi)

a = len(liste1)
b = (a+1) / 2
print(b)
```

Figure 2. Python code that provides the total count

The code in Figure 2, in general, takes a number from the user and calculates its square. Then, it computes the divisors of this square (list) and the values obtained by adding the square to the number (list1). For example, if the user enters 3, the square of this number will be 9, and list will contain 1 and 9, while list1 will contain 4 and 12. Next, it takes the length of list1 (a), adds one to this length (a+1), and divides it by 2 ((a+1)/2). It then prints the resulting value, b, to the screen.

Expressing a Unit Fraction as the Sum of Two Other Unit Fractions

In this stage, an algorithm developed to express selected unit fractions as the sum of two other unit fractions, along with examples of these sums obtained using the algorithm, will be presented.

Code 2. The ways in which a unit fraction can be written as the sum of two unit fractions were explained in the previous section. The generalization obtained at this stage has been transformed into the code presented in Figure 3 to provide a practical application.

```python
sayi = int(input("hangi sayıyının karesini çarpanlarına ayıralım? > "))

liste = []
liste1 = []
sirali_ikili = []
kare = sayi * sayi
for i in range(1, kare + 1):
    if kare % i == 0:
        print(str(i) + " * " + str(kare/i) + " = " + str(kare))
        liste.append(i)
        liste1.append(i + sayi)

print(liste)
print(liste1)

while len(liste1) > 1:
    sirali_ikili.append((liste1.pop(0), liste1.pop(-1)))

sirali_ikili.append((liste1[0], liste1[0]))

print(sirali_ikili)
```

Figure 3. Python code used to find the total list

The code in Figure 3 starts by taking an integer input from the user. It then calculates the square of the number entered by the user. It computes the divisors of the square number and the paired versions of these divisors. Two lists are used for this operation: one stores the divisors (list), and the other stores the sum of the divisors with the number taken from the user (list1). The lists containing the divisors and the sums of the divisors are printed to the screen. Next, for each element in the list1 list, the code iteratively takes an element from the beginning and the end of the list to create a paired couple, which is then added to a list named "paired_pairs." This process continues until all elements in the list1 list are consumed. Finally, the last element in the list1 list is paired with itself, and this pair is also added to the "paired_pairs" list.

Example

Let's express the unit fraction 1/2 as the sum of two other unit fractions.

For this, we'll use the generalization mentioned in the first stage to find the number of different sums that can be written.

Since our denominator is 2, the number of different sums that can be obtained is:

Number of different sums = (⟦C(2)⟧^2+1)/2. Here ⟦C(2)⟧^2= C(4)=3, {1,2,4} (3+1)/2=2. So, there should be 2 different sums. Now let's find these sums. We will use the 2nd algorithm developed for this purpose. Our number to create the denominator is: 2 Square of our number: 4 Factors of 4: {1,2,4} Increased by 2 for each factor: {3,4,6}

Pairs matched using the rainbow method from smallest to largest: (3,6) and (4,4). Let's use these pairs as the denominator. In this case, the unit fraction sums that give 1/2 are: 1/2=1/4+1/4 and 1/2=1/3+1/6

Example 2. Let's write unit fractions from 1/3 to 1/10 as the sum of two unit fractions using the developed algorithm.

1/3 can be written as 1/6 + 1/6 and 1/3 can also be written as 1/4 + 1/9 (can be written in 2 ways)

1/4 can be written as 1/8 + 1/8, 1/4 can also be written as 1/5 + 1/20, and 1/4 can also be written as 1/6 + 1/12 (can be written in 3 ways)

1/5 can be written as 1/10 + 1/10 and 1/5 can also be written as 1/6 + 1/30 (can be written in 2 ways)

1/6 can be written as 1/12 + 1/12, 1/6 can also be written as 1/10 + 1/15, 1/6 can also be written as 1/9 + 1/18, 1/6 can also be written as 1/8 + 1/24, and 1/6 can also be written as 1/7 + 1/42 (can be written in 5 ways)

1/7 can be written as 1/14 + 1/14 and 1/7 can also be written as 1/8 + 1/56 (can be written in 2 ways)

1/8 can be written as 1/16 + 1/16, 1/8 can also be written as 1/12 + 1/24, 1/8 can also be written as 1/10 + 1/40, and 1/8 can also be written as 1/9 + 1/70 (can be written in 4 ways)

1/9 can be written as 1/18 + 1/18, 1/9 can also be written as 1/12 + 1/36, and 1/9 can also be written as 1/10 + 1/90 (can be written in 3 ways)

1/10 can be written as 1/20 + 1/20, 1/10 can also be written as 1/15 + 1/30, 1/10 can also be written as 1/14 + 1/35, 1/10 can also be written as 1/12 + 1/60, and 1/10 can also be written as 1/11 + 1/110 (can be written in 5 ways)

Results and Discussion

The first result of our project on unit fractions is that a unit fraction can be written in as many different ways as half the number of positive divisors plus one of its denominator. This generalization was reached in the study and its mathematical proof was provided. Furthermore, for practical purposes, this generalization was formulated as an algorithm. The second result obtained from the research is the development of an algorithm showing how a unit fraction can be written as the sum of two unit fractions. With this algorithm, desired sums can be easily found. One of the application areas of the algorithms proposed for the sum of unit fractions is decorations made with regular polygons. For example, in all decorations that can be created using three regular polygons, the following condition must be met: In decorations created with regular polygons having “a” and “b” sides, and another polygon having “e” sides, it is necessary to determine how many different ways a fraction can be written as the sum of 3 or more unit fractions.

The sum of unit fractions can be used in various fields, including algorithm analysis and designing data structures in computer science. Especially when analyzing the complexity of data structures and algorithms, metrics related to the sum of unit fractions, such as time and space complexity, are important. These metrics are used to evaluate the performance of an algorithm or data structure and make decisions. This aligns with the findings of Gökoğlu’s (2017) metaphor study on algorithm perception in programming education. In other words, the use of the sum of unit fractions in this field helps in developing efficient and effective computer programs.

Another application area of developed algorithms is in engineering and industrial design. Particularly in areas such as material cutting, part assembly, and product design, the sum of fractions is important for precise measurements. In these areas, calculating the sum of unit fractions is necessary for correctly assembling materials and parts and ensuring the suitability of products. This aligns with the findings of Demir’s (2022) study on examining the reflection process of mathematical modeling on life. The use of the sum of unit fractions is a fundamental element in engineering and industrial design.

Another area where the project can be applied is in architecture and art, particularly in ornamental and decorative processes for aesthetic purposes. According to Aktaş (2022), in his study on the varieties of symmetry in decorative art, decorations are generally created by arranging geometric shapes, and mathematical calculations related to unit fractions play an important role in the design of structures. In this respect, this study can be considered as a significant contribution to the field.

The focus of this project is to investigate and develop new methods that can be used in solving mathematical problems. Researchers interested in gaining a deeper understanding of the methods used for adding fractions can focus on the advantages of these methods. Determining how many different ways a unit fraction can be written as the sum of two unit fractions has been an important point in the project. It would be appropriate to examine and develop this diversity with different methods. The developed rule in the project has been explained in detail, including how it was obtained. Additionally, it will be important to conduct studies on how the accuracy of this rule was tested and how it can be used in real-world applications.

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