Physics of the Dark Universe 44 (2024) 101457

Contents lists available at ScienceDirect

Physics of the Dark Universe

journal homepage: www.elsevier.com/locate/dark

Full length article

# Exploring non-perturbative corrections in thermodynamics of static dirty black holes

Saheb Soroushfar [a], Behnam Pourhassan [b,c,d,e,f], İzzet Sakallı [g,*]

[a] Department of Physics, College of Sciences, Yasouj University, 75918-74934, Yasouj, Iran
[b] School of Physics, Damghan University, Damghan, 3671641167, Iran
[c] Center for Theoretical Physics, Khazar University, 41 Mehseti Street, Baku, AZ1096, Azerbaijan
[d] Centre of Research Impact and Outcome, Chitkara University, Rajpura 140401, Punjab, India
[e] Physics Department, Istanbul Technical University, Istanbul, 34469, Turkey
[f] Canadian Quantum Research Center, 204-3002 32 Ave Vernon, BC V1T 2L7, Canada
[g] Physics Department, Eastern Mediterranean University, Famagusta 99628, North Cyprus via Mersin 10, Turkey

ARTICLE INFO

*Keywords:*
Non-perturbative correction
Black hole thermodynamics
Quantum work

ABSTRACT

This research delves into an extensive exploration of the thermodynamic characteristics exhibited by a contaminated black hole subject to a uniform electric field, within the theoretical framework of the Einstein-Nonlinear Electrodynamics (ENE)-dilaton theory. The investigation encompasses a thorough analysis of diverse thermodynamic facets, encompassing heat capacity, Helmholtz free energy, and internal energy. Through this comprehensive examination, valuable insights are provided into the distinctive behavior of the black hole when subjected to the influence of the electric field. Moreover, our study embarks on an exploration of the nuanced interplay between quantum effects and the thermodynamic profile, with a particular focus on scrutinizing the quantum-corrected entropy. This approach allows for a deeper understanding of the intricate relationship between quantum mechanics and the thermodynamic attributes exhibited by the system. By doing so, we aim to illuminate the non-perturbative corrections inherent in this intricate system, thereby contributing to a holistic comprehension of the altered thermodynamics characterizing dirty black holes within the confines of the specified theoretical framework. In essence, this research endeavors to uncover the subtleties of the modified thermodynamic landscape governing black holes tainted by external factors, specifically within the context of the ENE-dilaton theory. The outcomes of this study promise to extend our understanding of the intricate interactions within such complex systems, offering valuable insights into the non-perturbative corrections that manifest in their thermodynamic behavior.

## 1. Introduction

The field of black hole research has captivated and proven vital to the realm of theoretical physics and astrophysics [1,2]. Black holes are enigmatic objects that form when massive stars collapse under their gravity, creating a region in space where gravity is so strong that nothing, not even light, can escape its pull. These mysterious entities originate from the implosion of colossal stars and showcase extraordinary thermodynamic traits. The groundbreaking endeavors of Bekenstein and Hawking [3–6] established the bedrock principles of black hole thermodynamics [7], revealing that these entities possess temperature, entropy, and energy resembling those of thermodynamic systems, similar to ordinary thermodynamic systems. This idea was revolutionary because it treated black holes as thermodynamic objects rather than just gravitational entities. Thus, it is understood that black holes emit thermal radiation, which is known as Hawking radiation [8–14]. The temperature of this radiation, called Hawking temperature, is inversely proportional to the black hole's mass in the Schwarzschild family [15]. This temperature is incredibly low for massive black holes, making it hard to detect in practice. However, it has significant implications for the understanding of the universe's evolution and the connection between general relativity and quantum mechanics. It is also worth noting that especially in the non-asymptotically flat black holes, the temperature may remain constant, as being independent of the mass throughout the Hawking radiation. Such a phenomenon can occur during an isothermal process [16–18].

Entropy, another important thermodynamic property, measures the disorder or randomness present in a system. As mentioned earlier, Bekenstein and Hawking [19] found that black holes possess entropy

* Corresponding author.
*E-mail addresses:* soroush@yu.ac.ir (S. Soroushfar), b.pourhassan@du.ac.ir (B. Pourhassan), izzet.sakalli@emu.edu.tr (İ. Sakallı).

https://doi.org/10.1016/j.dark.2024.101457
Received 19 January 2024; Received in revised form 3 March 2024; Accepted 3 March 2024
Available online 6 March 2024

due to direct thermal fluctuations, as opposed to being attributed to their volume [20,21]. This finding was revolutionary because it suggested a link between the macroscopic properties of black holes and the microscopic world of quantum mechanics [22]. The field of black hole thermodynamics has expanded our understanding of the fundamental laws of physics. It has led to significant advancements in topics such as the holographic principle, quantum gravity, and the information paradox [23–25]. Researchers continue to explore this fascinating field, uncovering new insights into the nature of black holes and their role in the universe.

As our understanding of fundamental physics progressed, it became clear that the classical description of black hole thermodynamics may require modifications to fully encapsulate the underlying quantum gravitational effects and other exotic phenomena [26]. Various theories beyond Einstein's general relativity have been proposed [27–31] including complete reviews on modified gravity [32,33], encompassing novel fields and interactions that can potentially influence black hole thermodynamics. One such theory is the ENE-dilaton theory [34], which extends general relativity by incorporating nonlinear electrodynamics and a dilaton field [35]. In this scenario, the thermodynamic properties of black holes with nonlinear electrodynamics have been extensively explored in the literature [36–38]. In the presence of a uniform electric field, this theory introduces intriguing modifications to the properties of black holes, including their thermodynamic quantities. Furthermore, black holes in the context of this theory are often referred to as "dirty" due to the presence of additional fields and interactions beyond the vacuum solutions of classical general relativity [39,40].

It is widely known that in any thermodynamic system, there are thermal fluctuations occurring at a quantum level. These fluctuations contribute to the system's entropy, along with a logarithmic term that arises from a perturbative correction [41]. These perturbative corrections are more significant at scales larger than the Planck scale. However, at the Planck scale and smaller, non-perturbative corrections dominate [42]. This means that in thermodynamic systems like black holes, both perturbative and non-perturbative corrections play a role in determining the entropy [43,44]. One consequence of this is that an exponential term is added to modify the black hole area entropy through non-perturbative analysis. This correction appears in all quantum theories of gravity. The impact of this correction is negligible when the black hole has a large horizon radius, but becomes significant when the black hole size becomes extremely small.

In a recent article of Mazharimousavi [45], the author has successfully addressed key aspects of Einstein's gravity coupled with square-root-a nonlinear electrodynamics and a dilaton field: an ENE-dilaton theory. In that work, field equations have been precisely solved, yielding a unique black hole solution defined by two significant physical parameters: mass and dilaton field parameters. Notably, while the latter represents a constant characterizing the dilaton, the former is an integration parameter. This black hole is non-asymptotically flat (NAF) and exhibits singularity at its center, coinciding with the location of an electric charge. The electric field is radially symmetric and uniform, maintaining a constant electromagnetic invariant. One notable contribution of that study is the determination of the quasi-local conserved mass, denoted as $M_{QL}$, which was obtained by using the Brown-York formalism [46] since the ADM mass is not applicable to the NAF black holes. Remarkably, in the Schwarzschild limit, this newly defined mass coincides with the ADM mass of the Schwarzschild black hole. The article also explores the thermal stability of the black hole, revealing that it exhibits thermal stability under specific conditions ($0 < b^2 < 1$ or $2 < \eta^2$; for more details, we refer the reader to Ref. [45]), as evidenced by positive Hawking temperature and heat capacity. This finding underscores the existence of "dirty" black holes surrounded by normal matter fields, challenging the notion of black holes forming in empty space.

Our focus in this article delves into the non-perturbative corrections to the thermodynamics of dirty black holes of the ENE-dilaton theory [45]. We investigate key thermodynamic quantities of those dirty black holes such as heat capacity, Helmholtz free energy, and internal energy to uncover the effects of the additional fields and interactions introduced by the theory. Additionally, we consider the realm of quantum corrections and examine the modified entropy of these black holes, shedding light on the interplay between quantum effects and classical thermodynamics. So, we provide a comprehensive analysis of the thermodynamic aspects of black holes within the considered theory. Also, we study the intricate connection between quantum effects and black hole entropy, unveiling the quantum-corrected (QC) aspects of thermodynamic behavior. By examining the non-perturbative corrections to thermodynamic quantities, we aim to contribute to a deeper understanding of the nature of dirty black holes in the context of the ENE-dilaton theory.

The paper is organized as follows: In Section 2, we introduce the dirty black hole spacetime and highlight some of its distinctive features. In Section 3, we explore the theoretical framework of thermodynamics in dirty black hole geometry and examine heat capacity, Helmholtz free energy, and internal energy. In Section 4, we present the QC-entropy for the dirty black hole within the context of quantum work. Finally, in Section 5, we summarize our results and discuss potential avenues for further research in this intriguing field.

## 2. Features of dirty black holes of ENE-dilaton theory

In this section, the properties of the metric of the dirty black holes supported by a uniform electric field in the ENE-dilaton theory are studied. The action used in the ENE-dilaton theory is as follows [45]

$$\mathcal{I} = \int d^4x \left[\mathcal{R} - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + e^{-2b\psi}\mathcal{L}(\mathcal{F})\right], \tag{2.1}$$

where $b \neq 0$ is a free dilaton parameter, $\mathcal{R}$ is the Ricci scalar, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the electromagnetic invariant, $\mathcal{L}(\mathcal{F}) = \alpha\sqrt{-\mathcal{F}}$ and $\alpha$ is a dimensionful constant parameter. In the limit as $b \to \infty$, action (2.1) tends to revert to the Einstein-dilaton case. Varying the action concerning the metric tensor, dilaton scalar field, and gauge potential results in the Einstein field equation, dilaton field equation, and nonlinear electrodynamics-dilaton equation, respectively, [45]:

$$\mathcal{R}_\mu^\nu = 2\partial_\mu\psi\partial^\mu\psi + \frac{\alpha e^{-2b\psi}}{\sqrt{-\mathcal{F}}}F_{\mu\lambda}F^{\upsilon\lambda}, \tag{2.2}$$

$$\nabla_\mu\nabla^\mu\psi(r) = \frac{\alpha b e^{-2b\psi}}{2}\sqrt{-\mathcal{F}}, \tag{2.3}$$

$$d\left(\frac{e^{-2b\psi}}{\sqrt{-\mathcal{F}}}\tilde{F}\right) = 0, \tag{2.4}$$

where $\tilde{F}$ is the dual field two-form of $F$. At this point, a reader may question the significance of the chosen theory and, most importantly, the theory of non-linear electrodynamics considered in this paper can be thought as an ill-defined theory because the Lagrangian in Eq. (2.1) contains the square root of $\sqrt{-\mathcal{F}}$. To clarify this issue, let us recall that the pure electric or magnetic Born–Infeld (BI) nonlinear electrodynamics [47] is described by

$$L = b^2\left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2}}\right), \tag{2.5}$$

in which $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$. While the weak field limit of the theory ($\frac{\mathcal{F}}{b^2} \to 0$) is the linear Maxwell electrodynamics i.e.,

$$L \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.6}$$

and its strong field limit $\left(\frac{\mathcal{F}}{b^2} \to \infty\right)$ yields

$$L \sim \sqrt{\mathcal{F}}, \tag{2.7}$$

up to a constant coefficient. Therefore, the square-root model is not supposed to be considered instead of the linear electrodynamics theory but it is a model for the strong fields. Namely, $\sqrt{\mathcal{F}}$ corresponds to the strong field regime of the BI nonlinear electrodynamics theory, and hence there is no way to recover Maxwell's linear electrodynamics theory, which is in the weak field regime of the BI theory. Furthermore, as highlighted in the paper of Mazharimousavi [45], $b = 0$ ought to be excluded since it does not satisfy Einstein's equations. Similar studies can be seen in the literature from the pioneering works of G. 't Hooft [48] and H. B. Nielsen and P. Olesen [49]. In the sequel, important studies for the applications of the same model have been published by Guendelman and his colleagues [50–55], mainly on the confinement of quarks, and their followers like Mazharimousavi, Halilsoy, and Övgün [56–58] and references therein. Therefore, this ENE-dilaton model has a solid foundation and whence, in such a configuration, either there is a strong magnetic field or a strong electric field, one can set $L \sim \sqrt{\pm\mathcal{F}}$ [50–58].

Based on Eqs. (2.2)–(2.4), the metric for a static, spherically symmetric black hole spacetime having the dilaton field $\psi = \frac{1}{b}\ln r$ was expressed as [45]:

$$ds^2 = -\eta^2\left(1-\left(\frac{r_+}{r}\right)^{\eta^2}\right)r^{\frac{2}{b^2}}dt^2 + \frac{\eta^2}{1-\left(\frac{r_+}{r}\right)^{\eta^2}}dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{2.8}$$

where $r_+$ represents the event horizon of a black hole described by the above spacetime and $\eta^2 = \frac{b^2+1}{b^2} > 1$ is the dilaton parameter. Furthermore, Eq. (2.8) can be re-expressed as follows:

$$ds^2 = -(1-\frac{2M}{\rho^{\eta^2}})\rho^{2(\eta^2-1)}d\tau^2 + (1-\frac{2M}{\rho^{\eta^2}})^{-1}d\rho^2 + \frac{\rho^2}{\eta^2}(d\theta^2+\sin^2\theta d\varphi^2), \tag{2.9}$$

where $\rho = \eta r$, $M = \frac{(\eta r_+)^{\eta^2}}{2}$, and $\tau = \eta^{1-\frac{2}{b^2}}t$.

The Hawking temperature of the dirty black hole (2.9) can be computed with the aid of a timelike Killing vector ($\chi^\mu$) and whence the surface gravity ($\kappa$):

$$T_H = \frac{\kappa}{2\pi} = \frac{\nabla_\mu\chi^\mu\nabla_\nu\chi^\nu}{2\pi}\Bigg|_{r=r_+} = \frac{\eta^2}{4\pi}r_+^{\eta^2-2}, \tag{2.10}$$

and when the black hole area law [3] is applied, the black hole's entropy can be determined as

$$S_{BH} = \pi r_+^2. \tag{2.11}$$

It is also worth noting that the NAF structure of metric (2.8) admits the quasilocal mass [46] as $M_{QL} = \frac{r_+^{\eta^2}}{2}$ in which $\eta^2$ is related to the background. Thus, the first law of thermodynamics of the dirty black hole is satisfied as:

$$dM_{QL} = T_H dS_{BH}. \tag{2.12}$$

It is consistent with the following Smarr–Gibbs–Duhem relation,

$$\eta^2 M_{QL} = 2T_H S_{BH}, \tag{2.13}$$

which is indeed an integrated formula relating the black hole mass to the temperature and entropy. In the case of $b \to \infty$ ($\eta^2 = 1$) we recover the ordinary Schwarzschild results. The validity of the first law of thermodynamics is significant because it connects the change in quasi-local mass $M_{QL}$ to the changes in entropy and temperature. This implies that the thermodynamic behavior of these non-asymptotically flat dirty black holes still obeys the fundamental relation between heat, entropy, and energy. The form of the first law highlights the fact that the quasi-local mass plays the analogous role to internal energy for these solutions. Its changes correspond to exchange of heat defined by the Hawking temperature and black hole entropy. Therefore, Eq. (2.12) puts the exotic dirty black holes on equal thermodynamic footing with classical black holes.

Fig. 1 shows the behavior of the black hole temperature as a function of the horizon radius $r_+$. Different curves are plotted for various values of the parameter $\eta$, including $\eta^2 = 1$ which corresponds to the Schwarzschild case (dotted blue line).

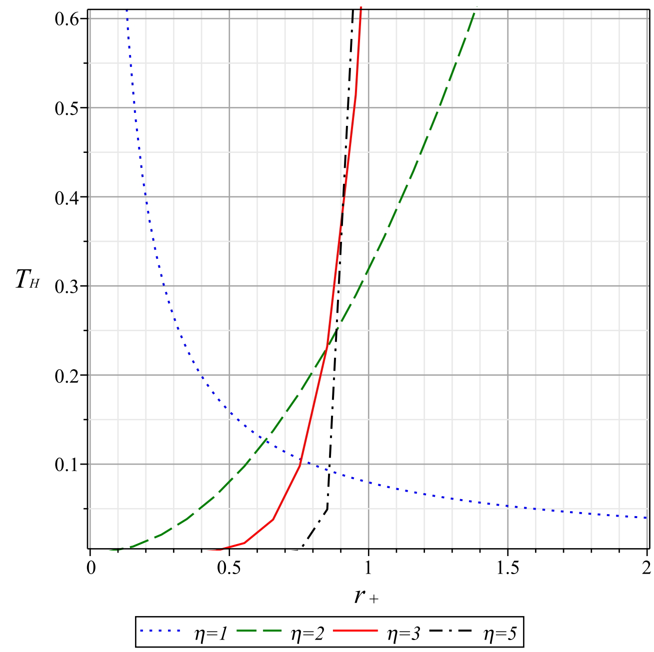



**Fig. 1.** Black hole temperature versus the horizon radius $r_+$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

As can be seen, the temperature starts from an extremely large positive value but decays rapidly, approaching zero asymptotically as the horizon radius increases. This is the expected qualitative behavior for most black holes — the temperature is inversely related to the mass.

More specifically, for the dirty black hole solutions studied here, the temperature is proportional to $r_+^{\eta^2-2}$. So for higher values of $\eta$, the initial peak temperature is lower and the subsequent decay is more gradual. The $\eta^2 = 1$ (Schwarzschild) case decays the fastest.

At very small horizon radius close to zero, the curves for $\eta \neq 1$ remain finite while the Schwarzschild temperature diverges. This highlights a key difference between the dirty black holes and classical vacuum solutions.

The positivity of the temperature curves indicates that these dirty black holes can be in stable thermal equilibrium for certain ranges of the parameters. This has important consequences for the thermodynamic properties studied in later sections. Overall, Fig. 1 provides insight into how the additional fields and interactions introduced in the ENE-dilaton theory qualitatively modify the temperature profile of the black holes across different mass scales. The non-asymptotically flat, non-vacuum nature of these solutions leads to distinct behavior.

## 3. Thermodynamics

In this section, we will study the thermodynamics of a dirty black hole supported by a uniform electric field in the ENE-dilaton theory by applying quantum corrections to the entropy.

### 3.1. Exponential correction

Researchers have used two types of quantum corrections: perturbative [59–64] and non-perturbative [65–69]. These corrections have been applied to analyze and investigate the thermodynamic properties of various black holes. It is well-known that the entropy of a system (denoted by $S_0$) is related to the number of measurable microstates ($\Omega$).

This relation can be expressed in units of Boltzmann constant ($k_B = 1$) as follows:

$$S_0 \equiv S_{BH} = \ln \Omega. \tag{3.1}$$

However, there exist other unmeasurable microstates contributing to entropy ($S_{\rm micro}$). The probability of encountering such states is inversely proportional to $\Omega$, where the corresponding entropy is also proportional to the probability. Therefore,

$$S_{\rm micro} \propto \frac{1}{\Omega}. \tag{3.2}$$

Consequently, we can express the total entropy as a sum of two components:

$$S = S_0 + S_{\rm micro}. \tag{3.3}$$

By combining Eqs. (3.1) and (3.2), we derive the relationship

$$S_{\rm micro} = e^{-S_0}. \tag{3.4}$$

Applying Stirling's approximation and utilizing statistical physics for a large total number $N$, the exponential quantum correction to black hole entropy is expressed as:

$$S = S_0 + e^{-S_0}, \tag{3.5}$$

which aligns with non-perturbative aspects of string theory [49,70] and is applicable to the Planckian regime of a black hole event horizon area. Subsequently, we utilize the non-perturbatively corrected entropy to examine the thermodynamic properties of the black hole that arise from these entropy modifications [71,72]:

$$S = S_0 + \lambda e^{-S_0}, \tag{3.6}$$

where $\lambda$ is the correction coefficient which is related to the proportionality constant in Eq. (3.2). So, we have

$$S = \pi r_+^2 + \lambda e^{-\pi r_+^2}, \tag{3.7}$$

which means that the last term of Eq. (3.7) affects the black hole thermodynamic quantities. It is important to highlight that the exponential correction becomes dominant as the black hole size decreases, primarily attributed to Hawking radiation at the Planck scale. Consequently, for large black holes, the correction coefficient is negligible, leading to the vanishing of the last term in Eq. (3.7). Conversely, for small black holes, the last term in Eq. (3.7) significantly influences the thermodynamics of the black hole.

Fig. 2 illustrates the behavior of black hole entropy with exponential correction plotted against the horizon radius for various values of the correction parameter. The case with $\lambda = 0$ represents the uncorrected scenario, consistent with the expected outcome from the standard Bekenstein–Hawking black hole entropy formula.

For a non-zero lambda, an exponential correction term is introduced, as outlined in Eq. (3.7). When lambda is positive, this correction leads the entropy curve to approach a maximum finite value asymptotically as $r_+$ becomes large. The magnitude of this entropy maximum decreases with an increase in lambda. This observation highlights the significance of non-perturbative quantum corrections for small black holes, imposing a fundamental constraint on the minimum attainable entropy. On the contrary, when $\lambda$ is negative, the exponential term leads to unbounded growth of entropy with increasing $r_+$. The more negative $\lambda$ becomes, the faster the entropy increases without bounds.

Fig. 2 illustrates the impact of exponential quantum corrections on black hole entropy, showcasing alterations in thermodynamic behavior, particularly at scales approaching the Planck length. The sign and magnitude of lambda play a crucial role in determining whether quantum effects impose fundamental constraints on minimum entropy or amplify the rates of entropy growth. These considerations bear profound implications for the stability and phase transitions of miniature black holes, where non-perturbative quantum gravity effects hold sway.

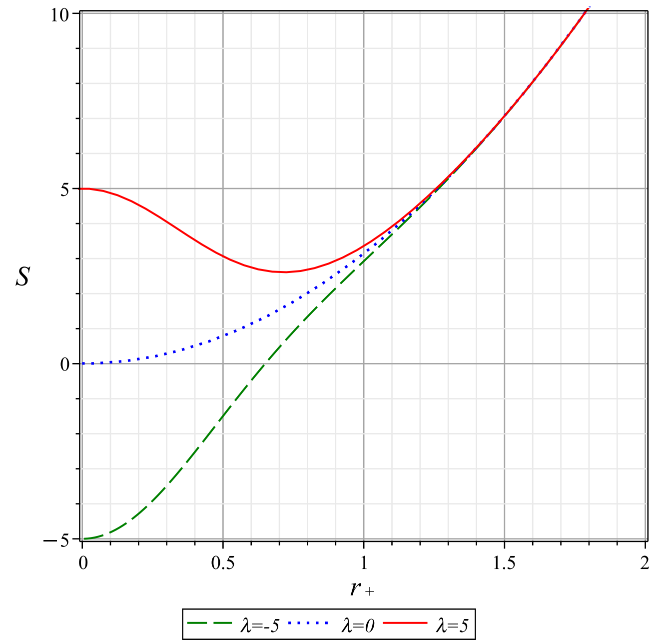



**Fig. 2.** Exponential corrected entropy versus the horizon radius $r_+$.

### 3.2. Heat capacity

One can compute the standard specific heat capacity of the dirty black hole as follows (see Eqs. (2.10) and (2.11)):

$$C_{BH} = T_H \frac{\partial S_{BH}}{\partial T_H} = \frac{2\pi r_+^2}{\eta^2 - 2}. \tag{3.8}$$

If both $T_H$ and $C_{BH}$ are positive, the black hole is considered thermally stable. Hence, when $\eta^2 - 2 > 0 \left(b^2 < 1\right)$, the black hole is thermally stable. When $\eta^2 = 2$, the Hawking temperature (2.10) remains constant, which corresponds to the infinite heat capacity. Similarly, one can compute the corrected heat capacity $\left(C = T_H \frac{\partial S}{\partial T_H}\right)$ by employing Eqs. (2.10) and (3.7) for the dirty black hole as follows

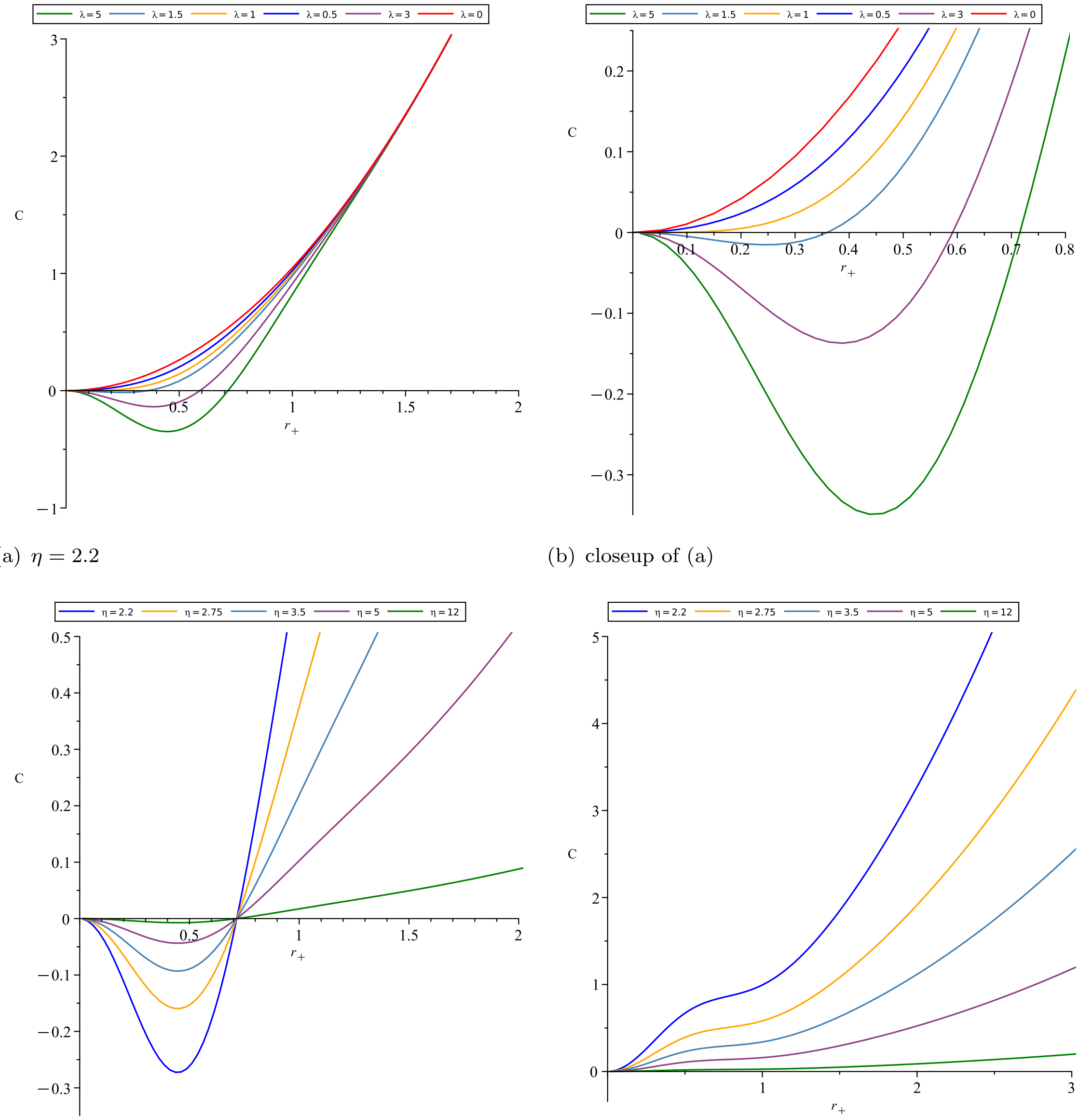
$$C = \frac{2\pi r_+^2 \left(\lambda\, e^{-\pi r_+^2} - 1\right)}{2 - \eta^2}, \tag{3.9}$$

which reduces to the original heat capacity $C_{BH}$ in the case of $\lambda = 0$.

Figs. 3(a)–(d) illustrate the behavior of a dirty black hole's heat capacity when supported by a uniform electric field according to the ENE-dilaton theory for various values of $\lambda$ and $\eta$. It can be observed from Figs. 3(a) and (b) that the original heat capacity ($\lambda = 0$) of this black hole is positive and has no phase transition. For the QC heat capacity, ($\lambda \neq 0$), and by increasing the values of $\lambda$, the heat capacity enters the negative region (unstable phase), then takes phase transition type one, and afterward it will be positive (stable) again. Hence, by reducing the black hole size due to Hawking radiation, the final stage of this black hole leads to instability (for the positive correction parameter).

Moreover, the behavior of the QC heat capacity for different values of the dilaton parameter ($\eta$) is shown in Figs. 3(c) and (d). In Fig. 3(c), we find that, for a certain positive value of $\lambda$, the system has phase transition type one, but for a certain negative value of $\lambda$, the system is in the stable phase and it has no phase transition [see Fig. 3(d)]. In this case, the final stage of the black hole is stable as well as uncorrected. In addition, from Fig. 3(c), it is clear that, by increasing the values of $\eta$, the QC heat capacity gradually tends to zero.

In any case, by looking at the aforementioned diagrams [Figs. 3(a)–(d)], analyzing them, and examining the variations in heat capacity




(a) $\eta = 2.2$ (b) closeup of (a)

(c) $\lambda = 5$ (d) $\lambda = -5$

**Fig. 3.** Variations of the original and the corrected heat capacity according to the horizon radius $r_+$.

for various values of $\lambda$ and $\eta$, as well as taking into account the phase transition that takes place for the heat capacity in the quantum correction mode, it can be deduced that a small black hole could be unstable due to quantum effects (assuming positive correction parameter), but a large black hole is in a thermodynamically stable phase. In the following subsections, we shall look at how the aforementioned quantum corrections affect the internal energy and Helmholtz free energy, among other thermodynamic variables.

### 3.3. Helmholtz free energy

In simple terms, Helmholtz free energy is a thermodynamic potential that combines internal energy and entropy, providing insight into a system's ability to perform work at constant temperature and volume [73]. When applied to black holes, it offers a unique perspective on their behavior. The Helmholtz free energy is used to understand the equilibrium between a black hole and its surrounding radiation, shedding light on the intricate interplay between mass, temperature, and entropy in these mysterious cosmic objects.

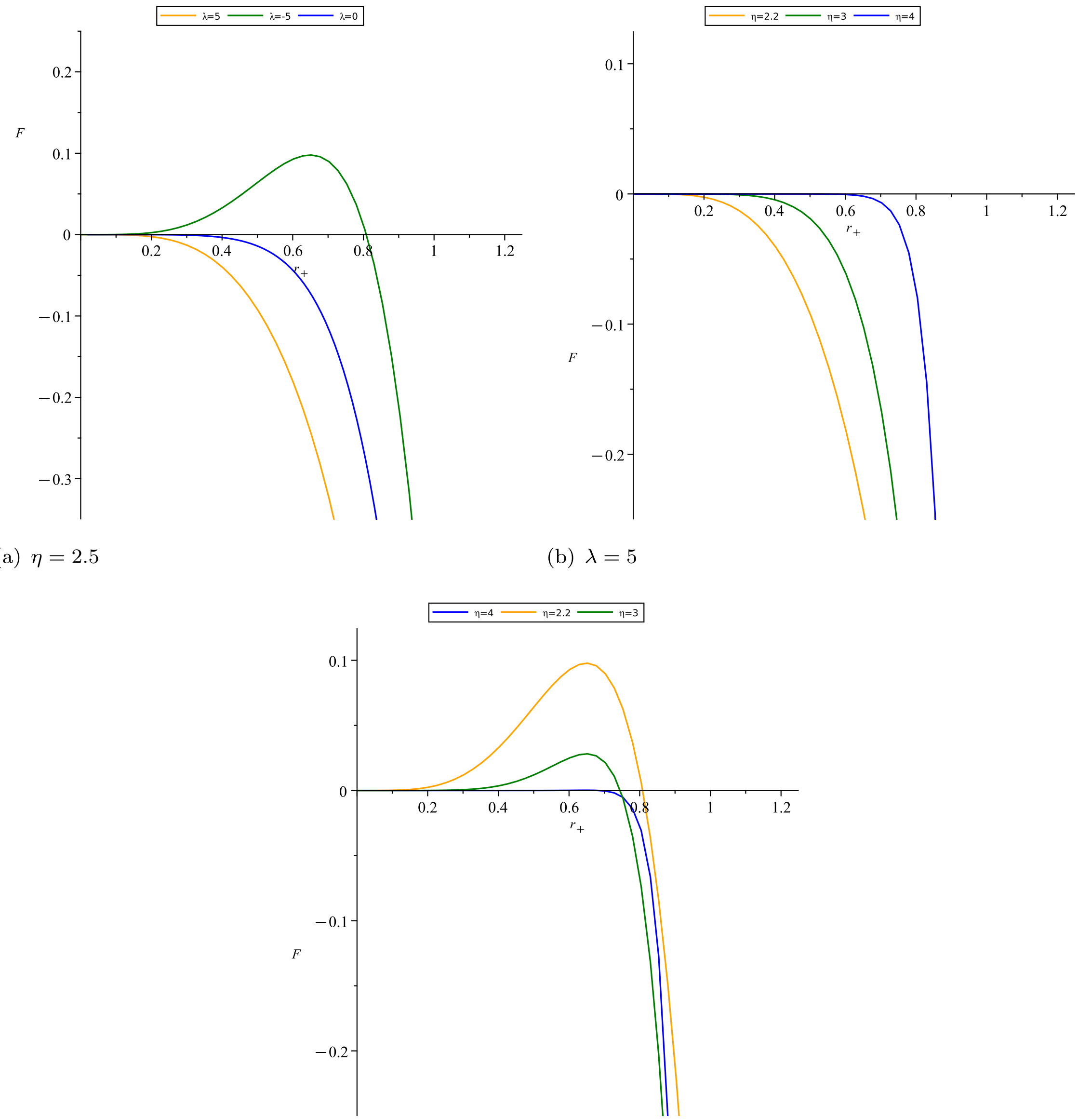
Here, the effects of quantum correction on the Helmholtz free energy, which can be fruitful in analyzing the stability and phase transition of a black hole, are investigated. The Helmholtz free energy is given by [74]:

$$F = -\int S dT. \tag{3.10}$$

So, by employing Eqs. (3.1), (3.7), and (3.10), one can get

$$F = -\frac{r_+^{\eta^2} \pi^{1-\frac{\eta^2}{4}} \mathrm{e}^{-\frac{\pi r_+^2}{2}} M_W\left(\frac{\eta^2}{4}, \frac{\eta^2}{4}+\frac{1}{2}, \pi r_+^2\right) \left(r_+^2\right)^{-\frac{\eta^2}{4}} \lambda}{\pi\left(\eta^2+2\right)} - \frac{\left(\lambda \eta^2 r_+^{\eta^2-2} + 2\lambda r_+^{\eta^2}\pi\right) \mathrm{e}^{-\pi r_+^2} + r_+^{\eta^2}\pi\left(\eta^2-2\right)}{4\pi}, \tag{3.11}$$

in which $M_W$ denotes the $WhittakerM(\mu, \nu, z)$ function and it can be defined in terms of the hypergeometric function [75,76]. In addition, the original Helmholtz free energy of the dirty black hole, in the case




(a) $\eta = 2.5$ (b) $\lambda = 5$

(c) $\lambda = -5$

**Fig. 4.** Variations of the Helmholtz free energy in terms of horizon radius $r_+$.

of $\lambda = 0$, can be written as follows

$$F_0 = -\frac{r_+^{\eta^2}\left(\eta^2 - 2\right)}{4}. \tag{3.12}$$

The behaviors of the Helmholtz free energy for different values of $\lambda$ and $\eta$ are plotted in Fig. 4.

Fig. 4(a) illustrates that, for a specific positive $\eta$ and $\lambda \geq 0$, the Helmholtz free energy is consistently negative, whereas it becomes positive for $\lambda < 0$. Additionally, in Fig. 4(b), we observe that, with a certain positive $\lambda$, the Helmholtz free energy remains negative across all values of $\eta$. Conversely, in Fig. 4(c), a specific negative $\lambda$ yields a positive Helmholtz free energy value, which subsequently decreases as $\eta$ values increase.

### 3.4. Internal energy

One of the most intriguing aspects of black holes is their thermodynamic behavior, which is described by analogies to classical thermodynamics. The concept of internal energy becomes essential in this context. In the study of black holes, internal energy refers to the energy contained within the black hole itself, often associated with the mass–energy equivalence principle, $E = Mc^2$. The process of Hawking radiation causes the black hole to lose mass over time and, consequently, its internal energy.

In this subsection, the effects of quantum correction on the internal energy is investigated. The general expression for the internal energy of a black hole is given by [77]

$$E = \int T_H dS. \tag{3.13}$$

Therefore, by using Eqs. (3.1), (3.7) and (3.13), we have

$$E = -\frac{2\left(e^{-\frac{\pi r_+^2}{2}} M_W\left(\frac{\eta^2}{4}, \frac{\eta^2}{4} + \frac{1}{2}, \pi r_+^2\right) \pi^{-\frac{\eta^2}{4}} \left(r_+^2\right)^{-\frac{\eta^2}{4}} \lambda + \frac{(\eta^2+2)\left(\lambda e^{-\pi r_+^2} - 1\right)}{2}\right) r_+^{\eta^2}}{2\eta^2 + 4}. \tag{3.14}$$

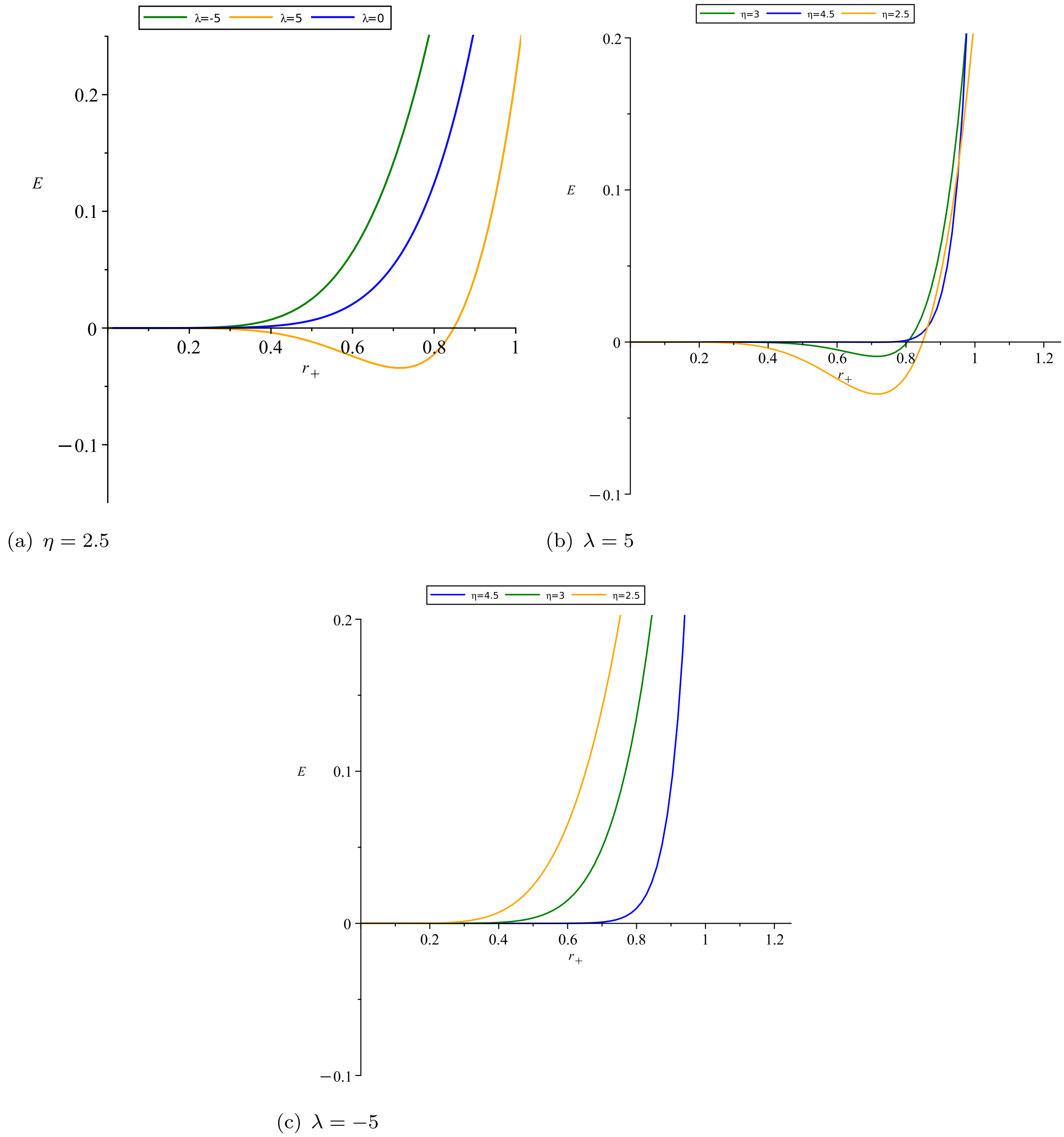



**Fig. 5.** Variations of the internal energy in terms of horizon radius $r_+$.

Therefore, the standard (when $\lambda = 0$) internal energy can be obtained as

$$E_0 = \frac{r_+^{\eta^2}}{2}. \tag{3.15}$$

The plots of the internal energy for different values of $\lambda$ and $\eta$, according to the horizon radius are depicted in Fig. 5.

Referring to Fig. 5(a), it becomes evident that, for a specific positive value of $\eta$ and when $\lambda \leq 0$, the internal energy consistently remains positive. However, when $\lambda > 0$, the internal energy takes on negative values. Furthermore, Fig. 5(b) reveals that, with a certain positive value of $\lambda$, the internal energy turns negative, and as the parameter $\eta$ increases, the internal energy gradually converges to zero. In addition, Fig. 5(c) illustrates that, for a particular negative value of $\lambda$, the internal energy remains positive across all values of $\eta$. At very small $r_+$, the internal energy trends negative when lambda is positive. This is because the exponential quantum corrections become very large, reducing the overall energy content of the tiny black hole. For negative lambda however, the internal energy remains positive regardless of size, as the corrections instead add to the energy. As $r_+$ increases, the exponential term quickly diminishes and in all cases the internal energy grows large and positive as expected classically. The higher the value of $\eta$, the more rapidly the internal energy converges to the uncorrected result. So in essence, Fig. 5 demonstrates how the sign of $\lambda$ determines whether quantum effects fundamentally limit the minimum internal energy for microscopic black holes or boost it to higher values. The dilaton parameter $\eta$ controls how rapidly the classical picture is recovered with increasing black hole scale. These intricate dependencies further highlight the significant role played by non-perturbative corrections in dictating the thermodynamic profile and content of tiny dirty black holes where semi-classical formulations break down. The plots advance our understanding of stability and phase transitions for these systems.

The key observation is that at very small horizon radius $r_+$, the exponential quantum corrections to the entropy and internal energy formulas become exceedingly large. Specifically, the term $\lambda e^{-\pi r_+^2}$ dominates as $r_+$ approaches zero. The sign of the correction parameter $\lambda$ determines whether this exponential term contributes positively or negatively. For a positive $\lambda$, the corrections render the overall internal energy expression negative when $r_+$ is tiny. Physically, this implies that quantum effects impose a fundamental limit on the minimum internal energy. In contrast, for a negative $\lambda$, the exponential corrections instead enhance the internal energy, maintaining it as positive even for microscopic black holes. Thus, negative $\lambda$ eliminates any theoretical lower bound on internal energy. As $r_+$ increases, the exponential term rapidly diminishes, and the standard classical picture is recovered, with internal energy growing large and positive. The higher the dilaton parameter $\eta$, the faster this convergence occurs.

Specifically, in Fig. 3(c), where the correction parameter $\lambda$ is positive, we observe a phase transition in the heat capacity, transitioning from negative (unstable) to positive (stable) as the horizon radius $r_+$ increases. This aligns with the behavior seen in Fig. 5(b) for positive $\lambda$; the internal energy is negative at small $r_+$ and rises back to positive values as $r_+$ increases. Therefore, the negative internal energy at microscopic scales corresponds directly to the black hole being in an unstable thermodynamic phase. Once $r_+$ increases sufficiently for the internal energy to become positive, stability is restored as the heat capacity turns positive as well.

In contrast, for negative $\lambda$ in Figs. 3(d) and 5(c), both the heat capacity and internal energy remain positive across all $r_+$, indicating a stable black hole phase even at the smallest scales.

## 4. Quantum work

This section delves into calculating the quantum work involved when the dirty black hole undergoes transitions between different states. This analysis is crucial because at microscopic scales comparable to the Planck length, quantum fluctuations and gravitational effects become very significant. It introduces corrections to classical thermodynamic concepts like entropy and free energy. Understanding how these quantum corrections affect the partition functions and probability distributions associated with microstates is key to modeling black hole emission/evaporation processes properly. The quantum work quantifies the relative weights of transitions in terms of modified free energy differences. So this section is motivated by the need to uncover how factors like the exponential correction to entropy alter equilibrium thermodynamics and statistical mechanics at a quantum gravitational level. This sheds light on the fundamental workings of tiny black holes where semi-classical physics breaks down. The concepts explored here promise to elucidate the role of quantum information theory and quantum gravity in dictating black hole stability, phase changes, and decay. The analysis lays the groundwork for assessing black hole information loss paradoxes while accounting for microscopic non-perturbative phenomena beyond the standard Hawking radiation theory.

The quantum-corrected entropy change of a dirty black hole subjected to a uniform electric field in the ENE-dilaton theory is expressed as [78]:

$$\Delta S = S_f - S_i, \tag{4.1}$$

where $S_i$ represents the initial entropy during the evolution and $S_f$ stands for the final entropy. Therefore, one can find out

$$\Delta S = \lambda\left(, \mathrm{e}^{-\pi r_{+f}^2} -, \mathrm{e}^{-\pi r_{+i}^2}\right) + \pi\left(r_{+f}^2 - r_{+i}^2\right). \tag{4.2}$$

Furthermore, to analyze the quantum work involved, it is beneficial to consider the change in the Helmholtz free energy. Thus, we can express the change in Helmholtz free energy ($\Delta F = F(r_{+f}) - F(r_{+i})$) and the quantum work $\left(e^{\frac{\Delta F}{T}}\right)$ [67] as follows:

$$\Delta F = -\frac{r_{+f}^{\eta^2} \pi^{1-\frac{\eta^2}{4}} \mathrm{e}^{-\frac{\pi r_{+f}^2}{2}} WM\left(\frac{\eta^2}{4}, \frac{\eta^2}{4}+\frac{1}{2}, \pi r_{+f}^2\right)\left(r_{+f}^2\right)^{-\frac{\eta^2}{4}} \lambda}{\pi\left(\eta^2+2\right)} - \tag{4.3}$$

$$\frac{\lambda\left(\frac{\eta^2 r_{+f}^{\eta^2-2}}{2} + r_{+f}^{\eta^2}\pi\right)\mathrm{e}^{-\pi r_{+f}^2} + \frac{\pi\left(\eta^2-2\right)\left(r_{+f}^{\eta^2} - r_{+i}^{\eta^2}\right)}{2}}{2\pi}, \tag{4.4}$$

and

$$W = e^{\frac{\Delta F}{T}} = \mathrm{e}^{-\frac{2 r_+^{-\eta^2+2}(\alpha 1+\alpha 2)}{(\eta^2+2)\eta^2}}, \tag{4.5}$$

by which

$$\alpha 1 = 2 r_{+f}^{\eta^2} \pi^{1-\frac{\eta^2}{4}} \mathrm{e}^{-\frac{\pi r_{+f}^2}{2}} M_W\left(\frac{\eta^2}{4}, \frac{\eta^2}{4}+\frac{1}{2}, \pi r_{+f}^2\right)\left(r_{+f}^2\right)^{-\frac{\eta^2}{4}} \lambda, \tag{4.6}$$

$$\alpha 2 = \left(\lambda\left(\frac{\eta^2 r_{+f}^{\eta^2-2}}{2} + r_{+f}^{\eta^2}\pi\right)\mathrm{e}^{-\pi r_{+f}^2} + \frac{\pi\left(\eta^2-2\right)\left(r_{+f}^{\eta^2} - r_{+i}^{\eta^2}\right)}{2}\right)\left(\eta^2+2\right). \tag{4.7}$$

Eq. (4.5) can be used to express quantum work in terms of the partition functions, denoted as $W = \frac{Z_f}{Z_i}$. The quantum work ($e^{\frac{\Delta F}{T}}$) for various values of $\lambda$ and $\eta$ can be visualized in relation to the horizon radius $r_+$, as shown in Fig. 6.

The distribution of partition functions for a black hole is influenced by the relative weights, or probabilities, associated with transitions between different states [69]. These weights are determined by the quantum work performed during the transition. Quantum work relies on the difference in equilibrium free energies between the initial and final states, which, in turn, depends on the microstates of the black hole. As the black hole emits radiation, understanding this process requires consideration of the average quantum work during the emission process [78]. The significance of quantum work becomes more pronounced at small scales, necessitating the incorporation of quantum gravitational corrections. It is evident from the curves in Fig. 6 that, at larger values of $r_+$, all curves coincide.

Fig. 6 shows how the quantum work, calculated as $e^{\frac{\Delta F}{T}}$, varies with the black hole horizon radius $r_+$ for different values of the correction parameters lambda and eta. In all cases, the quantum work starts from a small negative value at very low $r_+$ and initially rises rapidly before plateauing and converging at larger horizon radius. This profile reflects the complex dependence of the quantum work on the changes in free energy between initial and final black hole states, as described by Eqs. (4.5)–(4.7).

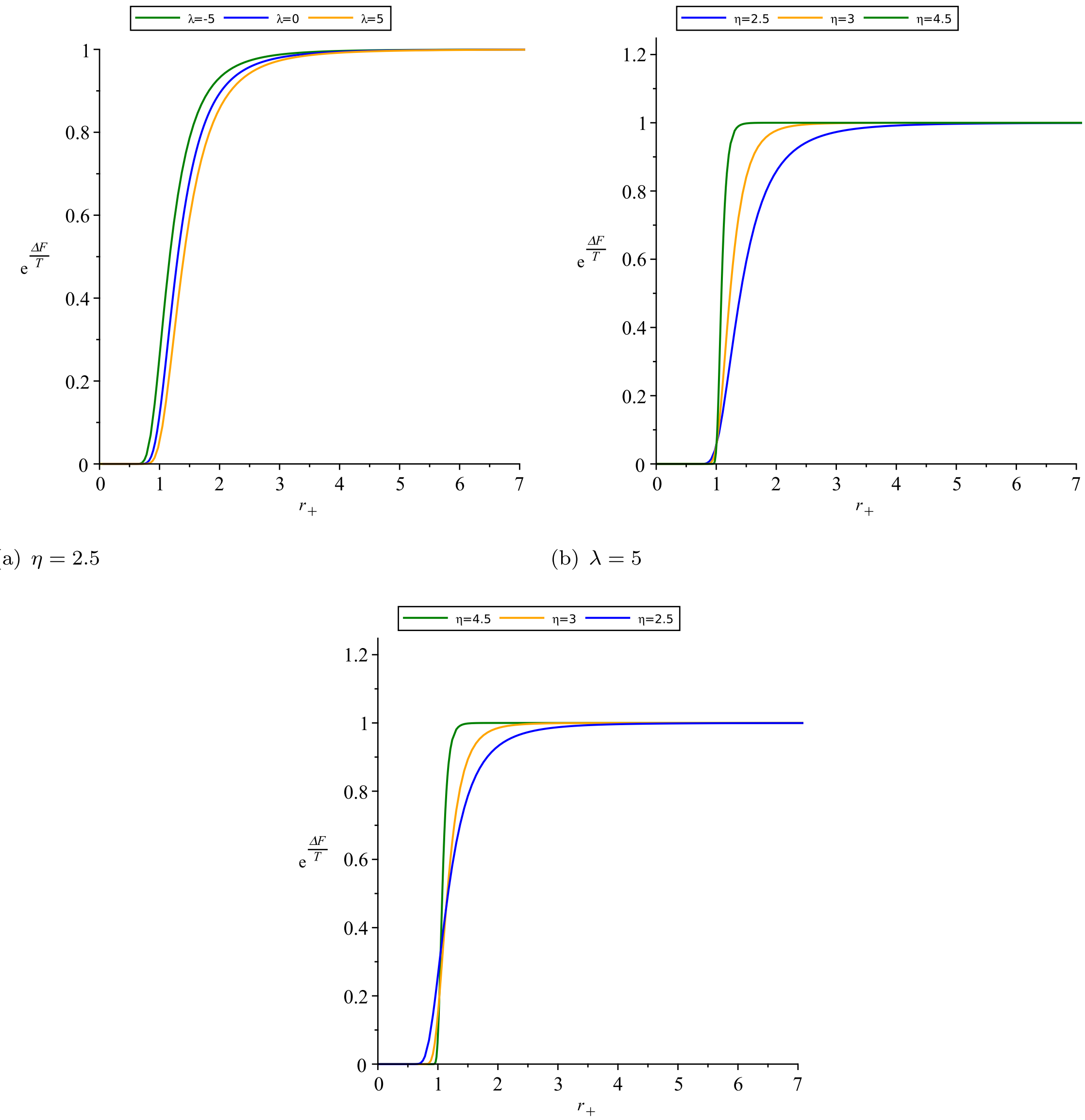
Specifically, at small $r_+$, the exponential corrections to the entropy and free energy are more pronounced, leading to larger differences between initial and final free energies as the black hole shrinks. This results in the rapidly increasing magnitude of quantum work. As $r_+$ increases, the black hole grows in size and the exponential corrections become negligible. Thus, the free energy change and corresponding quantum work starts to plateau. All curves converge at large enough $r_+$ when the corrections no longer contribute.

Comparing the plots, more positive lambda leads to higher overall values of quantum work, while more negative lambda suppresses it. Similarly, increasing the parameter eta boosts the quantum work. This further verifies the sensitive dependence on the exponential corrections.

In summary, Fig. 6 elucidates how incorporating microstate fluctuations and non-perturbative effects significantly alters predictions of the thermodynamic work involved in black hole transitions. This underscores the importance of quantum corrections in dictating black hole behavior. These corrections modify equilibrium free energies, subsequently affecting the calculation of quantum work. Therefore, it becomes imperative to integrate these modified expressions for free energies to accurately assess the impact of quantum gravitational corrections on the distribution of quantum work [69,78–80].

## 5. Conclusion

In a recent study, a novel solution has been introduced that involves a dirty/hairy black hole within the framework of ENE-dilaton




(a) $\eta = 2.5$

(b) $\lambda = 5$

(c) $\lambda = -5$

**Fig. 6.** Variations of quantum work ($e^{\frac{\Delta F}{T}}$) with respect to the horizon radius $r_+$.

theory [45]. This particular black hole is characterized by a uniform radial electric field and a singular dilaton scalar field, which is NAF and possesses a singularity at its core. This type of black hole is of significant interest due to its connections with charged black holes in string theory, where the hair/dilaton field is non-minimally coupled to electromagnetic fields. These connections have far-reaching implications in various areas, such as the AdS/CFT correspondence, which hints at a holographic duality with a quark–gluon plasma [81]. Consequently, there is a pressing need to delve into the thermodynamics of this intriguing black hole, especially at quantum scales, which is the central focus of the present paper.

The paper has commenced by providing a comprehensive overview of the dirty black hole sustained by a uniform electric field in the ENE-dilaton theory. Subsequently, we have derived the entropy of this particular dirty black hole, accounting for exponential corrections. We have also conducted an in-depth examination of various thermodynamic properties associated with this black hole. An analysis of the heat capacity reveals that the final stage of this black hole becomes unstable, assuming that the correction parameter $\lambda$ is positive, as anticipated. Furthermore, we have computed the quantum work employing free energy and investigated the influence of exponential corrections on it. Remarkably, it is observed that both non-perturbative and perturbative corrections exert a substantial impact, particularly as the black hole size (horizon radius) diminishes due to the process of Hawking radiation.

In light of these findings, future research avenues emerge from this article. First and foremost, it is imperative to explore the consequences of the black hole's instability in the late stages of its evolution, which could have profound implications for our understanding of gravitational dynamics. Additionally, further investigations into the holographic duality between this type of black hole and quark–gluon plasma, as suggested by the AdS/CFT correspondence, could yield valuable insights into the behavior of matter under extreme conditions. It is noteworthy that the same analysis can be applied to other entropy corrections arising from quantum considerations, such as Barrow entropy [82,83], or alternative considerations, as exemplified by Tsallis entropy [84] and Kaniadakis entropy [85]. Finally, the interplay between the dilaton field and thermodynamic properties of black holes remains a fertile ground for future research, particularly in the context of quantum gravity and string theory. These avenues promise exciting developments in our comprehension of the fundamental aspects of the universe.

## CRediT authorship contribution statement

**Saheb Soroushfar:** Writing – original draft, Visualization, Validation, Software, Investigation, Formal analysis. **Behnam Pourhassan:** Writing – original draft, Methodology, Investigation, Formal analysis. **İzzet Sakallı:** Writing – review & editing, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Izzet SAKALLI reports financial support was provided by Eastern Mediterranean University. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Acknowledgments

We express our sincere gratitude to the Editor and the anonymous Referee for their invaluable time, expertise, and constructive feedback, which greatly contributed to the enhancement of this manuscript. Their thoughtful comments and insightful suggestions have played a pivotal role in shaping the quality and clarity of our work. We also wish to convey our appreciation to Prof. Dr. S. H. Mazharimousavi for his insightful comments on Eqs. (2.5)–(2.7), and enriching discussions regarding the topic of dirty black holes. İ.S. would like to acknowledge networking support of COST Actions CA21106 and CA22113. He further thanks to TÜBİTAK, SCOAP3, and ANKOS for their support.

## References

[1] R.M. Wald, General Relativity, Chicago University Press, Chicago, 1984.
[2] V.P. Frolov, I.D. Novikov, Black Hole Physics: Basic Concepts and New Developments, Kluwer Academic Publishers, Dordrecht, 1998.
[3] J.D. Bekenstein, Phys. Rev. D 7 (1973) 2333.
[4] S.W. Hawking, Phys. Rev. Lett. 26 (1971) 1344.
[5] J.D. Bekenstein, Phys. Rev. D 9 (1974) 3292.
[6] S.W. Hawking, Commun. Math. Phys. 43 (1975) 199—220.
[7] G.W. Gibbons, S.W. Hawking, Phys. Rev. D 15 (1977) 2738–2751.
[8] S.W. Hawking, D.N. Page, Comm. Math. Phys. 87 (1983) 577.
[9] M.K. Parikh, F. Wilczek, Phys. Rev. Lett. 85 (2000) 5042–5045.
[10] S.W. Hawking, Phys. Rev. D 13 (1976) 191–197.
[11] I. Sakalli, A. Ovgun, Europhys. Lett. 118 (6) (2017) 60006.
[12] I. Sakalli, A. Ovgun, Europhys. Lett. 110 (1) (2015) 10008.
[13] I. Sakalli, M. Halilsoy, H. Pasaoglu, Astrophys. Space Sci. 340 (1) (2012) 155–160.
[14] I. Sakalli, O.A. Aslan, Astrophys. Space Sci. 361 (4) (2016) 128.
[15] S.W. Hawking, Nature 248 (1974) 30–31.
[16] S.H. Mazharimousavi, I. Sakalli, M. Halilsoy, Phys. Lett. B 672 (2009) 177–181.
[17] H. Pasaoglu, I. Sakalli, Internat. J. Theoret. Phys. 48 (2009) 3517–3525.
[18] İ. Sakalli, S. Kanzi, Turk. J. Phys. 46 (2) (2022) 51–103.
[19] J. Bekenstein, Scholarpedia 3 (10) (2008) 7375.
[20] A. Strominger, C. Vafa, Phys. Lett. B 379 (1996) 99–104.
[21] A. Strominger, J. High Energy Phys. 02 (1998) 009.
[22] M. Li, Classical Quantum Gravity 21 (2004) 3571–3578.
[23] S. Raju, Phys. Rep. 943 (2022) 1–80.
[24] H.Z. Chen, Z. Fisher, J. Hernandez, R.C. Myers, S.M. Ruan, J. High Energy Phys. 03 (2020) 152.
[25] M. Hotta J. Trevison and K. Yamaguchi.
[26] B.S. DeWitt, Phys. Rev. 160 (1967) 1113–1148.
[27] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rep. 513 (2012) 1–189.
[28] A. De Felice, S. Tsujikawa, Living Rev. Rel. 13 (2010) 3.
[29] E. Berti, E. Barausse, V. Cardoso, L. Gualtieri, P. Pani, U. Sperhake, L.C. Stein, N. Wex, K. Yagi, T. Baker, et al., Classical Quantum Gravity 32 (2015) 243001.
[30] Y.F. Cai, S. Capozziello, M. De Laurentis, E.N. Saridakis, Rep. Progr. Phys. 79 (10) (2016) 106901.
[31] E. Palti, Fortschr. Phys. 67 (6) (2019) 1900037.
[32] S. Nojiri, S.D. Odintsov, Phys. Rep. 505 (2011) 59–144.
[33] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rep. 692 (2017) 1–104.
[34] A. Sheykhi, S. Hajkhalili, Phys. Rev. D 89 (10) (2014) 104019.
[35] H.A. Gonzalez, M. Hassaine, C. Martinez, Source. Phys. Rev. D 80 (2009) 104008.
[36] G.G.L. Nashed, Emmanuel N. Saridakis, Classical Quantum Gravity 36 (13) (2019) 135005.
[37] G.G.L. Nashed, Emmanuel N. Saridakis, Phys. Rev. D 102 (2020) 124072.
[38] G.G.L. Nashed, Emmanuel N. Saridakis, J. Cosmol. Astropart. Phys. 05 (05) (2022) 017.
[39] M. Visser, Phys. Rev. D 48 (1993) 5697–5705.
[40] M. Visser, Phys. Rev. D 46 (1992) 2445–2451.
[41] S. Upadhyay, N. ul Islam, P.A. Ganai, J. Holography Appl. Phys. 2 (1) (2022) 25–48, http://dx.doi.org/10.22128/jhap.2021.454.1004.
[42] A. Chatterjee, A. Ghosh, Phys. Rev. Lett. 125 (2020) 041302.
[43] A.J.M. Medved, E.C. Vagenas, Modern Phys. Lett. A 20 (2005) 1723–1728.
[44] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 03 (2015) 074.
[45] S.H. Mazharimousavi, Eur. Phys. J. C 83 (5) (2023) 406; Eur. Phys. J. C 83 (7) (2023) 597, arXiv:2304.12935. [gr-qc].
[46] J.W. Brown, J.D. York Jr., Phys. Rev. D 47 (1993) 1407–1419.
[47] M. Born, L. Infeld, Nature 132 (3348) (1933) 1004.1.
[48] G. 't Hooft, Nucl. Phys. B Proc. Suppl. 121 (2003) 333–340.
[49] H.B. Nielsen, P. Olesen, Nuclear Phys. B 57 (1973) 367–380.
[50] P. Gaete, E. Guendelman, Phys. Lett. B 640 (2006) 201–204.
[51] P. Gaete, E. Guendelman, E. Spallucci, Phys. Lett. B 649 (2007) 218–224.
[52] E. Guendelman, A. Kaganovich, E. Nissimov, S. Pacheva, Phys. Lett. B 704 (2011) 230–233; erratum: Phys. Lett. B 705 (2011) 545-545.
[53] E. Guendelman, M. Vasihoun, Classical Quantum Gravity 29 (2012) 095004.
[54] E. Guendelman, E. Nissimov, S. Pacheva, AIP Conf. Proc. 2075 (1) (2019) 090030.
[55] E.I. Guendelman, G.J. Olmo, D. Rubiera-Garcia, M. Vasihoun, Phys. Lett. B 726 (2013) 870–875.
[56] S.H. Mazharimousavi, M. Halilsoy, Phys. Lett. B 710 (2012) 489–492.
[57] A. Övgün, Phys. Lett. B 820 (2021) 136517.
[58] S.H. Mazharimousavi, Phys. Scripta 98 (1) (2023) 015305.
[59] B. Pourhassan, M. Faizal, J. High Energ. Phys. 2021 (2021) 50.
[60] B. Pourhassan, S. Dey, S. Chougule, M. Faizal, Classical Quantum Gravity 37 (2020) 135004.
[61] B. Pourhassan, S. Upadhyay, H. Saadat, H. Farahani, Nuclear Phys. B 928 (2018) 415.
[62] B. Pourhassan, M. Faizal, Z. Zaz, A. Bhat, Phys. Lett. B 773 (2017) 325.
[63] B. Pourhassan, A. Ovgun, I. Sakalli, Int. J. Geom. Methods Mod. Phys. 17 (2020) 2050156.
[64] S. Upadhyay, S. Soroushfar, R. Saffari, Modern Phys. Lett. A 36 (29) (2021) 2150212, arXiv:1801.09574. [gr-qc].
[65] S.W. Wei, Y.X. Liu, R.B. Mann, Phys. Rev. Lett. 123 (2019) 071103.
[66] A. Chatterjee, A. Ghosh, Phys. Rev. Lett. 125 (2020) 041302.
[67] S. Soroushfar, H. Farahani, S. Upadhyay, Phys. Dark Univ. 42 (2023) 101272.
[68] B. Pourhassan, H. Aounallah, M. Faizal, S. Upadhyay, S. Soroushfar, Y.O. Aitenov, S.S. Wani, J. High Energy Phys. 05 (2022) 030, arXiv:2201.11073. [hep-th].
[69] B. Pourhassan, S.S. Wani, S. Soroushfar, M. Faizal, J. High Energy Phys. 10 (2021) 027, arXiv:2102.03296 [hep-th].
[70] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 1503 (2015) 074.
[71] B. Pourhassan, M. Dehghani, M. Faizal, S. Dey, Classical Quantum Gravity 38 (2021) 105001.
[72] B. Pourhassan, J. Stat. Mech. 2021 (2021) 073102.
[73] J. Sadeghi, B. Pourhassan, M. Rostami, Phys, Rev. D 94 (6) (2016) 064006.
[74] B.P. Dolan, Classical Quantum Gravity 28 (2011) 125020.
[75] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1972.
[76] Y. Luke, The Special Functions and their Approximations, vol. 1, Academic Press, 1969.
[77] S.A. Hayward, Class, Quant. Grav. 15 (1998) 3147–3162.
[78] B. Pourhassan, M. Atashi, H. Aounallah, S.S. Wani, M. Faizal, B. Majumder, Nuclear Phys. B 980 (2022) 115842.
[79] B. Pourhassan, R. Delgado Campos, arXiv:2205.00238. [hep-th].
[80] B. Pourhassan, M. Faizal, J. High Energy Phys. 10 (2021) 050.
[81] J. Sadeghi, B. Pourhassan, J. High Energy Phys. 12 (2008) 026.
[82] Emmanuel N. Saridakis, Phys. Rev. D 102 (2020) 123525.
[83] S. Maity, P. Rudra, J. Holography Applications in Physics 2 (1) (2022) 1–12, http://dx.doi.org/10.22128/jhap.2022.464.1012.
[84] Emmanuel N. Saridakis, K. Bamba, R. Myrzakulov, Fotios K. Anagnostopoulos, J. Cosmol. Astropart. Phys. 1812 (12) (2018) 012.
[85] N. Drepanou, A. Lymperis, Emmanuel N. Saridakis, K. Yesmakhanova, Eur. Phys. J. C 82 (5) (2022) 449.