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Inverse Scattering Method via Riemann–Hilbert Problem for Nonlinear Klein–Gordon Equation Coupled with a Scalar Field

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A class of negative order Ablowitz–Kaup–Newell–Segur nonlinear evolution equations are obtained by applying the Lax hierarchy of the first order linear system of three equations. The inverse scattering problem on the whole axis is examined in the case where linear system becomes the classical Zakharov–Shabat system consists of two equations and admits a real anti-symmetric potential. Referring to these results, the N-soliton solutions for the integro-differential version of the nonlinear Klein–Gordon equation coupled with a scalar field are obtained by using the inverse scattering method via the Riemann–Hilbert problem.

1. Introduction

A completely integrable nonlinear equation of mathematical physics is one which has a Lax representation, or, more precisely, can be solved by the inverse scattering method via a Riemann–Hilbert (RH) problem, the classic examples being the Korteweg–de Vries, sine-Gordon and nonlinear Schrödinger equations.[1)] This approach is closely connected with the nonlinear Fourier method and one of the most powerful techniques to study integrable equations and particularly generate soliton solutions.[2)] A few integrable equations, including the multiple wave interaction equations, the general coupled nonlinear Schrödinger equations, the Harry Dym equation and the generalized Sasa-Satsuma equation have been studied in Refs. 3–5 by solving the associated RH problems.

It is applied the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy to derive soliton solutions of some integrable models by the inverse scattering method via RH problem. Recently, many integrable hierarchies of soliton equations have been extended to hierarchies of a negative order AKNS equation by many authors.[6–9)] This gives an useful necessary extension for complete integrability, which is applied to investigate the integrability of certain generalizations of the Klein–Gordon equations, some model nonlinear wave equations of nonlinear Klein–Gordon equation coupled with a scalar field.

Consider the nonlinear Klein–Gordon equation coupled with a field v, in the following form:[10)]

$$\begin{cases} u_{\varkappa\varkappa} - u_{\tau\tau} - u + 2u^3 + 2vu = 0, \\ v_\varkappa - v_\tau - 4uu_\tau = 0. \end{cases} \tag{1.1}$$

In the case $v \neq 0$, this equation is integrable since it admits the same bilinear form with the well-known sine-Gordon equation.[10)]

The coupled nonlinear Klein–Gordon equations are analyzed for their integrability properties in Ref. 11 where the Hirota bilinear form is identified, from which one-soliton solutions are derived. Then, the results are generalized to the two, three and N-coupled Klein–Gordon equations in Refs. 12 and 13. Another direct method for traveling wave solutions of coupled nonlinear Klein–Gordon equations is employed in Ref. 14.

The equation (1.1) becomes the following negative first order equation:

$$r_{tx} = 2r\partial_x^{-1}[(r^2)_t] + r \tag{1.2}$$

by the change of variables $\varkappa = \frac{x+4t}{2}$, $\tau = \frac{4t-x}{2}$, elimination of v in second equation under the assumption that the scalar field v tends to zero at infinity and by the substitution $r = \frac{1}{2}u$, where $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x.

Our aim in this paper is to find the soliton solutions of (1.2) by the inverse scattering method via RH problem. The inverse scattering method is the most important discovery in the theory of soliton. It provides us alternatively show the complete integrability of the nonlinear evolution equation. This method also enables to solve the initial value problem for nonlinear evolution equation (1.2). Shortly we call the equation (1.2) the coupled Klein–Gordon (CKG) equation in future.

The CKG equation also closely relates to the Maxwell–Bloch (MB) equation where the detuning function is a Dirac delta. In this spirit, we recommend that more references should be added about MB equation, such as the Lax pair for the MB system was first found in Ref. 15 by using the results of Refs. 16 and 17. The N-soliton solutions for the MB equations by applying inverse scattering method via the RH approach are in papers.[18,19)]

The brief outline of the paper is the followings. In Sect. 2, we find that the CKG possesses a Lax pair of the negative order AKNS equation. It is shown that the auxiliary systems corresponding to CKG is classical Zakharov–Shabat (ZS) system with real and anti-symmetric potential. Then, in Sect. 3, we obtained the RH problem of inverse scattering problem for the general ZS equation on the whole line. In Sect. 4, we give solution of RH problem under some conditions on the zeros of the elements of scattering matrix for the general ZS system. This conditions are violated in the case of ZS system with real and anti-symmetric potential. In last section, the N-soliton solutions of the CKG equation are obtained by the solution of RH problem.

2. Negative First Order AKNS Equations

Consider the spectral problem for 3×3 linear system

$$\begin{bmatrix} \varphi_{1x} \\ \varphi_{2x} \\ \varphi_{3x} \end{bmatrix} = X(p,q) \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \tag{2.1}$$

where $X(p,q) = i \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 \\ q_1 & \alpha_2\lambda & 0 \\ q_2 & 0 & \alpha_2\lambda \end{bmatrix}$ with λ is a nonzero eigenvalue, φ_1, φ_2, and φ_3 are linearly independent eigenfunctions, $i^2 = -1$, α_1 and α_2 are real constants with $\alpha_1 - \alpha_2 = \alpha < 0$, $p_1 = p_1(x,t)$, $p_2 = p_2(x,t)$, $q_1 = q_1(x,t)$, and $q_2 = q_2(x,t)$ are the rapidly decreasing at infinity complex valued coefficients.

The auxiliary spectral problem described as follows:

$$\begin{bmatrix} \varphi_{1t} \\ \varphi_{2t} \\ \varphi_{3t} \end{bmatrix} = T(p,q) \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \tag{2.2}$$

where $T(p,q) = \begin{bmatrix} a & b & c \\ d & e & f \\ k & l & m \end{bmatrix}$ and a, b, c, d, e, f, k, l, and m are scalar functions, independent of φ_1, φ_2, and φ_3.

From (2.1) and (2.2), the zero curvature equation $X_t - T_x + [X,T] = 0$ yields

$$\begin{aligned} a_x &= ip_1 d + ip_2 k - iq_1 b - iq_2 c, \qquad f_x = iq_1 c - ip_2 d, \\ b_x &= i\alpha\lambda b - ip_1\alpha + ip_1 e + ip_2 l + ip_{1t}, \qquad l_x = iq_2 b - ip_1 k, \\ c_x &= i\alpha\lambda c + ip_1 f - ip_2\alpha + ip_2 m + ip_{2t}, \qquad m_x = iq_2 c - ip_2 k, \\ d_x &= -i\alpha\lambda d + iq_1 a - iq_1 e - iq_2 f + iq_{1t}, \\ k_x &= -i\alpha\lambda k - iq_1 l + iq_2 a - iq_2 m + iq_{2t}, \\ e_x &= iq_1 b - ip_1 d, \end{aligned} \tag{2.3}$$

where $\alpha = \alpha_1 - \alpha_2$. Let the following transformations be applied to the system (2.3):

$$a = \frac{A(x,t)}{\lambda}, \quad b = \frac{B(x,t)}{\lambda}, \quad c = \frac{C(x,t)}{\lambda},$$
$$d = \frac{D(x,t)}{\lambda}, \quad e = \frac{E(x,t)}{\lambda}, \quad f = \frac{F(x,t)}{\lambda},$$
$$k = \frac{K(x,t)}{\lambda}, \quad l = \frac{L(x,t)}{\lambda}, \quad m = \frac{M(x,t)}{\lambda}.$$

As a result the following equations are obtained:

$$\begin{aligned} &A_x = ip_1 D + ip_2 K - iq_1 B - iq_2 C, \qquad E_x = iq_1 B - ip_1 D, \\ &B_x = ip_1 E - ip_1 A + ip_2 L, \quad B = -\frac{1}{\alpha} p_{1t}, \quad L_x = iq_2 B - ip_1 K, \\ &C_x = ip_1 F + ip_2 M - ip_2 A, \quad C = -\frac{1}{\alpha} p_{2t}, \quad M_x = iq_2 C - ip_2 K, \\ &D_x = iq_1 A - iq_1 E - iq_2 F, \quad D = \frac{1}{\alpha} q_{1t}, \\ &K_x = -iq_1 L + iq_2 A - iq_2 M, \quad K = \frac{1}{\alpha} q_{2t}, \\ &F_x = iq_1 C - ip_2 D. \end{aligned} \tag{2.4}$$

The following negative first order AKNS equations are obtained for important cases of spectral problem (2.1).

Proposition 1. *If the coefficients of (2.1) satisfies the properties $p_1 = q_1$ and $p_2 = q_2$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= p_1[2\partial_x^{-1}(p_1^2)_t + \partial_x^{-1}(p_2^2)_t + c_1] + p_2[\partial_x^{-1}(p_1 p_2)_t + c_2], \\ p_{2tx} &= p_2[\partial_x^{-1}(p_1^2)_t + 2\partial_x^{-1}(p_2^2)_t + c_3] + p_1[\partial_x^{-1}(p_1 p_2)_t + c_4], \end{aligned} \tag{2.5}$$

where c_k, $k = 1,2,3,4$ are arbitrary constants and $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x.

Proof. It is clearly seen that $A_x = -E_x - M_x$, $B = -D$, $C = -K$, and $F_x = L_x$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t}, \quad C = -\frac{1}{\alpha} p_{2t},$$
$$E_x = -\tfrac{i}{\alpha}(p_1^2)_t, \quad F_x = -\tfrac{i}{\alpha}(p_1 p_2)_t, \quad M_x = -\tfrac{i}{\alpha}(p_2^2)_t,$$
$$B_x = ip_1(2E + M) + ip_2 F,$$
$$C_x = ip_1 F + ip_2(E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.5), where $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x. □

Proposition 2. *If the coefficients of (2.1) satisfies the properties $p_1 = -q_1$ and $p_2 = -q_2$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= -p_1[2\partial_x^{-1}(p_1^2)_t + \partial_x^{-1}(p_2^2)_t + c_1] \\ &\quad - p_2[\partial_x^{-1}(p_1 p_2)_t + c_2], \\ p_{2tx} &= -p_2[\partial_x^{-1}(p_1^2)_t + 2\partial_x^{-1}(p_2^2)_t + c_3] \\ &\quad - p_1[\partial_x^{-1}(p_1 p_2)_t + c_4], \end{aligned} \tag{2.6}$$

where c_k, $k = 1,2,3,4$ are arbitrary constants and $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x.

Proof. It is clearly seen that $A_x = -E_x - M_x$, $B = D$, $C = K$, and $F_x = L_x$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t}, \quad C = -\frac{1}{\alpha} p_{2t},$$
$$E_x = \tfrac{i}{\alpha}(p_1^2)_t, \quad F_x = \tfrac{i}{\alpha}(p_1 p_2)_t, \quad M_x = \tfrac{i}{\alpha}(p_2^2)_t,$$
$$B_x = ip_1(2E + M) + ip_2 F,$$
$$C_x = ip_1 F + ip_2(E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.6), where $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x. □

Proposition 3. *If the coefficients of (2.1) satisfies the properties $p_1 = q_1^*$ and $p_2 = q_2^*$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= p_1[2\partial_x^{-1}(|p_1^2|)_t + \partial_x^{-1}(|p_2^2|)_t + c_1] \\ &\quad + p_2[\partial_x^{-1}(p_1 p_2^*)_t + c_2], \\ p_{2tx} &= p_2[\partial_x^{-1}(|p_1^2|)_t + 2\partial_x^{-1}(|p_2^2|)_t + c_3] \\ &\quad + p_1[\partial_x^{-1}(p_1^* p_2)_t + c_4], \end{aligned} \tag{2.7}$$

where c_k, $k = 1, 2, 3, 4$ are arbitrary constants and $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x.

Proof. It is clearly seen that $A_x = -E_x - M_x$, $B = -D^*$, $C = -K^*$, and $F_x = -L_x^*$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t}, \quad C = -\frac{1}{\alpha} p_{2t},$$

$$E_x = -\tfrac{i}{\alpha}(|p_1^2|)_t, \quad F_x = -\tfrac{i}{\alpha}(p_1^* p_2)_t, \quad M_x = -\tfrac{i}{\alpha}(|p_2^2|)_t,$$

$$B_x = ip_1(2E + M) - ip_2 F^*,$$

$$C_x = ip_1 F + ip_2(E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.7), where $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x. □

Proposition 4. *If the coefficients of (2.1) satisfies the properties $p_1 = -q_1^*$ and $p_2 = -q_2^*$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= -p_1[2\partial_x^{-1}(|p_1^2|)_t + \partial_x^{-1}(|p_2^2|)_t + c_1] \\ &\quad - p_2[\partial_x^{-1}(p_1 p_2^*)_t + c_2], \\ p_{2tx} &= -p_2[\partial_x^{-1}(|p_1^2|)_t + 2\partial_x^{-1}(|p_2^2|)_t + c_3] \\ &\quad - p_1[\partial_x^{-1}(p_1^* p_2)_t + c_4], \end{aligned} \tag{2.8}$$

where c_k, $k = 1, 2, 3, 4$ are arbitrary constants and $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x.

Proof. It is clearly seen that $A_x = -E_x - M_x$, $B = D^*$, $C = K^*$, and $F_x = -L_x^*$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t}, \quad C = -\frac{1}{\alpha} p_{2t},$$

$$E_x = \tfrac{i}{\alpha}(|p_1^2|)_t, \quad F_x = \tfrac{i}{\alpha}(p_1^* p_2)_t, \quad M_x = \tfrac{i}{\alpha}(|p_2^2|)_t,$$

$$B_x = ip_1(2E + M) - ip_2 F^*,$$

$$C_x = ip_1 F + ip_2(E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.8), where $\partial_x^{-1} = \int_x^\infty dx$ is indefinite integral with respect to x. □

The following corollary of Proposition 2 is valid.

Corollary 1. *In the case $p_1 = p_2 = p$ and $c_1 = c_2 = c_3 = c_4 = -\frac{1}{4}$ this nonlinear evolution equation has the form*

$$p_{tx} = -4p \int_x^{+\infty} (p^2)_t \, dx + p.$$

This equation becomes CKG equation by the substitution $p = \frac{ir}{\sqrt{2}}$ with real valued $r(x)$. The spectral problem (2.1) for this equation is the classical Zakharov–Shabat system with real and anti-symmetric potential $r = r(x)$.

Remark 1. Referring to the results for inverse scattering problem for the Manakov system (see Appendix of the paper[20]), the soliton solutions for negative order AKNS equation (2.7) can be examined similarly to the pair of nonlinear Schrodinger equations which is made in Ref. 20 for two-dimensional stationary self-focusing of electromagnetic waves.

3. Riemann–Hilbert Problem for Zakharov–Shabat System

In this section, we recall the necessary results from Ref. 21 on inverse scattering problem for classical Zakharov–Shabat system:

$$\begin{cases} u_{1x} = -i\mu u_1 - r u_2, \\ u_{2x} = r u_1 + i\mu u_2 \end{cases} \tag{3.1}$$

with real coefficient $r = r(x)$.

If $-p_1 = -p_2 = q_1 = q_2 = p$ are taken in (2.1), we obtain

$$\begin{cases} \varphi_{1x} = i\alpha_1 \lambda \varphi_1 - ip(\varphi_2 + \varphi_3), \\ \varphi_{2x} = ip\varphi_1 + i\alpha_2 \lambda \varphi_2, \\ \varphi_{3x} = ip\varphi_1 + i\alpha_2 \lambda \varphi_3. \end{cases}$$

This system becomes

$$\begin{cases} v_{1x} = i\alpha_1 \lambda v_1 + ipv_2, \\ v_{2x} = 2ipv_1 + i\alpha_2 \lambda v_2, \end{cases}$$

by the substitutions $\sqrt{2}\varphi_1 = u_1$, $\varphi_2 + \varphi_3 = -u_2$, $-\alpha_1 = \alpha_2 = \beta$, $\mu = \beta\lambda$, and $p = \frac{ir}{\sqrt{2}}$ transforms this system to classical ZS system with real and anti-symmetric potential.

As is shown in previous section, the CKG equation admits the Lax representation which the components are ZS system with real and anti-symmetric potential and the another component is the form:

$$\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} \dfrac{A(x,t)}{\mu} & \dfrac{B(x,t)}{\mu} \\ \dfrac{C(x,t)}{\mu} & -\dfrac{A(x,t)}{\mu} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.2}$$

where the compatibility condition becomes

$$A_x = -r(C + B),$$

$$B_x = 2Ar, \quad C_x = 2Ar,$$

$$r_t = -2iB, \quad r_t = -2iC.$$

This system produce the equation

$$A = i \int_x^{+\infty} r r_t \, dx + c_0 \tag{3.3}$$

and

$$r_{tx} = 2r \int_x^{+\infty} (r^2)_t \, dx - 4ic_0 r$$

where is CKG equation when $c_0 = \frac{i}{4}$.

We have seen that requering $r \to 0$ as $|x| \to \infty$ gives us a large class of equations with the property that $\frac{A(x,t)}{\mu} \to A_-(\mu)$, $\frac{B(x,t)}{\mu} \to 0$, $\frac{C(x,t)}{\mu} \to 0$ as $|x| \to \infty$. It is clear from (3.3) that $A_-(\mu) = \frac{i}{4\mu}$.

Consider more general spectral problems of the following form:

$$-iU_x = XU, \quad -iU_t = TU, \tag{3.4}$$

with

$$X = \mu\Lambda + P, \quad T = -\frac{h}{\mu}\Lambda + Q, \quad \mu \neq 0, \quad h = const > 0$$

where

$$\Lambda = \mathrm{diag}(-1, 1),$$

$$P = -\begin{bmatrix} 0 & ip \\ iq & 0 \end{bmatrix}, \quad Q \to 0 \text{ as } |x| \to \infty.$$

The cases $-p = q = r$, $h = \frac{1}{4}$, and $Q \to 0$ as $|x| \to \infty$ of the systems (3.4) yields the (3.1), (3.2).

In this section, we present the scattering and inverse scattering methods for the system (3.4) using the RH formulation. Impose the condition on the complex valued functions $q(x)$ and $p(x)$ that decay sufficiently rapidly as $|x| \to \infty$.

In the RH formulation, we treat U in the spectral problems (3.4) as a fundamental matrix. From (3.4), we note that, when $x, t \to \pm\infty$, one has the asymptotic behavior: $U \sim E = e^{i\mu\Lambda x - \frac{ih}{\mu}\Lambda t}$. This motivates us to

$$\Psi = Ue^{-i\mu\Lambda x + \frac{ih}{\mu}\Lambda t},$$

to have the canonical normalization for the associated RH problem:

$$\Psi \to I_2, \quad \text{when } x, t \to \pm\infty,$$

where $I_2 = \mathrm{diag}(1, 1)$. This way, the spectral problems in (3.4) equivalently lead to

$$\Psi_x = i\mu[\Lambda, \Psi] + \breve{P}\Psi, \tag{3.5}$$
$$\Psi_t = -\frac{ih}{\mu}[\Lambda, \Psi] + \breve{Q}\Psi,$$

where $\breve{P} = iP$ and $\breve{Q} = iQ$. Noting $tr(\breve{P}) = tr(\breve{Q}) = 0$, we have

$$\det\Psi = 1 \tag{3.6}$$

by Abel's formula.

Let us now consider the formulation of an associated RH problem with the variable x. In the scattering problem, we first introduce the matrix solutions $\Psi^{\pm}(x, \mu)$ of (3.5) with the asymptotic conditions

$$\Psi^{\pm} \to I_2, \quad \text{when } x \to \pm\infty, \tag{3.7}$$

respectively. The subscripts above refer to which end of the x-axis the boundary conditions are required. Then, by (3.6), we have $\det\Psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since $U^{\pm} = \Psi^{\pm}E$ are both solutions of (3.4), they must be linearly related, and so we can have

$$\Psi^{-}E = \Psi^{+}ES(\mu), \quad \mu \in \mathbb{R}, \quad \mu \neq 0 \tag{3.8}$$

where

$$S(\mu) = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \quad \mu \in \mathbb{R}, \quad \mu \neq 0$$

is the scattering matrix. Note that $\det(S(\mu)) = 1$ since $\det(\Psi^{\pm}) = 1$. Using the method of variation of parameters as well as the boundary condition (3.7), we can turn the x-part of (3.4) into the following Volterra integral equations for $\Psi^{\pm}$:

$$\Psi^{-}(\mu, x) = I_2 + \int_{-\infty}^{x} e^{i\mu\Lambda(x-y)}\breve{P}(y)\Psi^{-}(\mu, y)e^{i\mu\Lambda(y-x)}\,dy, \tag{3.9}$$

$$\Psi^{+}(\mu, x) = I_2 - \int_{\infty}^{x} e^{i\mu\Lambda(x-y)}\breve{P}(y)\Psi^{+}(\mu, y)e^{i\mu\Lambda(y-x)}\,dy.$$

Thus, $\Psi^{\pm}$ allows analytical continuations off the real axis $\mu \in \mathbb{R}$ ($\mu \neq 0$) as long as the integrals on their right hand sides converge. It is direct to see that the integral equation for the first column of Ψ^{-} contains only the exponential factor $e^{2i\mu(x-y)}$, which, due to $y < x$ in the integral, decays when μ is in the upper half-plane $\mathbb{C}^{+} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, and the integral equation for the second column of Ψ^{+} contains only the exponential factor $e^{-2i\mu(x-y)}$, which, due to $y > x$ in the integral, also decays when μ is in the upper half-plane $\mathbb{C}^{+}$. Thus, these two columns can be analytically continued to the upper half-plane $\mu \in \mathbb{C}^{+}$. Similarly, we find that the second column of Ψ^{-} and the first column of Ψ^{+} can be analytically continued to the lower half-plane $\mu \in \mathbb{C}^{-} = \{z \in \mathbb{C} | \operatorname{Im}(z) < 0\}$. Let us express

$$\Psi^{\pm} = (\Psi_1^{\pm}, \Psi_2^{\pm}),$$

that is, $\Psi_k^{\pm}$ stands for the kth column of $U^{\pm}$ $(1 \leq k \leq 2)$. Then the matrix solution

$$P^{+} = P^{+}(x, \mu) = (\Psi_1^{-}, \Psi_2^{+}) = \Psi^{-}H_1 + \Psi^{+}H_2$$

is analytic in $\mu \in \mathbb{C}^{+}$, and the matrix solution

$$(\Psi_1^{+}, \Psi_2^{-}) = \Psi^{+}H_1 + \Psi^{-}H_2$$

is analytic in $\mu \in \mathbb{C}^{-}$, where $H_1 = \mathrm{diag}(1, 0)$ and $H_2 = \mathrm{diag}(0, 1)$. In addition, from the Volterra integral equation (3.9), we find that

$$P^{+}(x, \mu) \to I_2, \quad \text{when } \mu \in \mathbb{C}^{+} \to \infty,$$

and

$$(\Psi_1^{+}, \Psi_2^{-}) \to I_2, \quad \text{when } \mu \in \mathbb{C}^{-} \to \infty.$$

Next we construct the analytic counterpart of P^{+} in the lower half-plane $\mathbb{C}^{-}$. Note that the adjoint equation of the x-part of (3.4) and the adjoint equation of (3.5) read as

$$i\widetilde{U}_x = \widetilde{U}X,$$

and

$$i\widetilde{\Psi}_x = \mu[\widetilde{\Psi}, \Lambda] + \widetilde{\Psi}P.$$

It is easy to see that the inverse matrices $\widetilde{U}^{\pm} = (U^{\pm})^{-1}$ and $\widetilde{\Psi}^{\pm} = (\Psi^{\pm})^{-1}$ solve these adjoint equations, respectively. If we express $\widetilde{\Psi}^{\pm}$ as follows:

$$\widetilde{\Psi}^{\pm} = \begin{bmatrix} \widetilde{\Psi}^{\pm,1} \\ \widetilde{\Psi}^{\pm,2} \end{bmatrix},$$

that is, $\widetilde{\Psi}^{\pm,k}$ stands for the kth row of $\widetilde{\Psi}^{\pm}$ $(1 \leq k \leq 2)$. Then by similar arguments, we can show that adjoint matrix solution

$$P^{-} = \begin{bmatrix} \widetilde{\Psi}^{-,1} \\ \widetilde{\Psi}^{+,2} \end{bmatrix} = H_1\widetilde{\Psi}^{-} + H_2\widetilde{\Psi}^{+} = H_1(\Psi^{-})^{-1} + H_2(\Psi^{+})^{-1}$$

is analytic for $\mu \in \mathbb{C}^{-}$, and the other matrix solution

$$\begin{bmatrix} \widetilde{\Psi}^{+,1} \\ \widetilde{\Psi}^{-,2} \end{bmatrix} = H_1\widetilde{\Psi}^{+} + H_2\widetilde{\Psi}^{-} = H_1(\Psi^{+})^{-1} + H_2(\Psi^{-})^{-1}$$

is analytic for $\mu \in \mathbb{C}^{+}$. In the same way we see that

$$P^{-}(x, \mu) \to I_2, \quad \text{when } \mu \in \mathbb{C}^{-} \to \infty,$$

$$\begin{bmatrix} \widetilde{\Psi}^{+,1} \\ \widetilde{\Psi}^{-,2} \end{bmatrix} \to I_2, \quad \text{when } \mu \in \mathbb{C}^+ \to \infty.$$

Now we have constructed two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. It is direct to see that on the real line, the two matrix functions P^+ and P^- related by

$$P^-(x,\mu)P^+(x,\mu) = G(x,\mu), \quad \mu \in \mathbb{R}, \quad \mu \neq 0 \tag{3.10}$$

where

$$G(x,\mu) = E(H_1 + H_2 S)(H_1 + S^{-1}H_2)E^{-1} \tag{3.11}$$

$$= E\begin{bmatrix} 1 & -s_{12} \\ s_{21} & s_{11}s_{22} - s_{12}s_{21} \end{bmatrix}E^{-1}.$$

Equations (3.10) and (3.11) are exactly the associated matrix RH problem we wanted to present. The asymptotics

$$P^\pm(x,\mu) \to I_2, \quad \text{when } \mu \in \mathbb{C}^\pm \to \infty, \tag{3.12}$$

provide the canonical normalization condition for the established RH problem.

To finish the direct scattering transform, we take the derivative of (3.8) with time t and use the vanishing conditions of the potentials, we can show that S satisfies

$$S_t = -\frac{ih}{\mu}[\Lambda, S], \quad \mu \neq 0,$$

which gives the time evolution of the scattering coefficients:

$$s_{11,t} = s_{22,t} = 0,$$
$$s_{12,t} = s_{12}(\mu,0)e^{-\frac{2iht}{\mu}},$$
$$s_{21,t} = s_{21}(\mu,0)e^{\frac{2iht}{\mu}}.$$

4. N-Soliton Solutions

The RH problem with zeros generate soliton solutions. The uniqueness of the associated RH problem (3.10) does not hold unless zeros of $\det P^+$ and $\det P^-$ in the upper and lower half-planes are specified and the kernel structures of $P^\pm$ at these zeros are determined. Following the definitions of $P^\pm$ as well as the scattering relation between Ψ^+ and Ψ^-, we find that

$$\det P^+(x,\mu) = s_{11}(\mu), \quad \det P^-(x,\mu) = s_{22}(\mu),$$

due to $\det S = 1$.

Theorem 1. *Suppose that s_{11} has zeros $\{\mu_k \in \mathbb{C}^+,\ 1 \le k \le N\}$, and s_{22} has zeros $\{\widehat{\mu}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$. Then the N-soliton solution to the system of (3.4) is*

$$p = -2i\sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\widehat{v}_{l,2},$$
$$q = 2i\sum_{k,l=1}^{N} v_{k,2}(M^{-1})_{kl}\widehat{v}_{l,1}, \tag{4.1}$$

where

$$v_k(x,t) = e^{i\mu_k\Lambda x - \frac{ih}{\mu_k}\Lambda t}v_{k,0}, \quad 1 \le k \le N,$$
$$\widehat{v}_k(x,t) = \widehat{v}_{k,0}e^{-i\widehat{\mu}_k\Lambda x + \frac{ih}{\widehat{\mu}_k}\Lambda t}, \quad 1 \le k \le N, \tag{4.2}$$

with column $v_{k,0}$ and row $\widehat{v}_{k,0}$, $1 \le k \le N$ arbitrary constant vectors, $v_k = (v_{k,1}, v_{k,2})^T$ and $\widehat{v}_k = (\widehat{v}_{k,1}, \widehat{v}_{k,2})$ and $M = (M_{kl})_{N\times N}$ is a square matrix whose entries are

$$M_{kl} = \frac{\widehat{v}_k v_l}{\mu_l - \widehat{\mu}_k}, \quad 1 \le k,\ l \le N.$$

Proof. For simplicity, we assume that zeros, μ_k and $\widehat{\mu}_k$, $1 \le k \le N$, are simple. Then, each of $\ker P^+(\mu_k)$, $1 \le k \le N$, contains only a single column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker P^-(\widehat{\mu}_k)$, $1 \le k \le N$, a row vector, denoted by $\widehat{v}_k$, $1 \le k \le N$:

$$P^+(\mu_k)v_k = 0, \quad \widehat{v}_k P^-(\widehat{\mu}_k) = 0, \quad 1 \le k \le N. \tag{4.3}$$

The RH problem (3.10) with the canonical normalization condition (3.12) and the zero structure (4.3) can be solved explicitly, and thus one can readily reconstruct the potential P as follows. Note that P^+ is a solution to the spectral problem (3.4). Therefore, as long as we expand P^+ at large μ as

$$P^+(x,\mu) = I_2 + \frac{1}{\mu}P_1^+(x) + O\left(\frac{1}{\mu^2}\right), \quad \mu \to \infty$$

inserting this expansion into (3.5) and comparing $O(1)$ terms lead to

$$\breve{P} = -i[\Lambda, P_1^+],$$

which implies that

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & 2(P_1^+)_{12} \\ -2(P_1^+)_{21} & 0 \end{bmatrix},$$

where $P_1^+ = ((P_1^+)_{kl})_{1\le k,\, l\le 2}$. Further, the potentials p and q, can be computed as

$$p = 2i(P_1^+)_{12}, \quad q = -2i(P_1^+)_{21}. \tag{4.4}$$

To obtain soliton solutions, we set $G = I_2$ in the RH problem (3.10). This can be achieved if we assume $s_{12} = s_{21} = 0$, $s_{11}s_{22} = 1$, which means that there is no reflection in the scattering problem. The solution to this specific RH problem can be given as follows:

$$P^+(\mu) = I_2 - \sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\widehat{v}_l}{\mu - \widehat{\mu}_l},$$
$$P^-(\mu) = I_2 + \sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\widehat{v}_l}{\mu - \mu_l}, \tag{4.5}$$

where $M = (M_{kl})_{N\times N}$ is a square matrix whose entries read

$$M_{kl} = \frac{\widehat{v}_k v_l}{\mu_l - \widehat{\mu}_k}, \quad 1 \le k,\ l \le N.$$

Noting that the zeros μ_k and $\widehat{\mu}_k$ are constants, i.e., space and time independent, we can easily find the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\widehat{v}_k(x,t)$, $1 \le k \le N$. For example, let us take the x-derivative of both sides of the equation $P^+(\mu_k)v_k = 0$. By using (3.5) and then $P^+(\mu_k)v_k = 0$, we get

$$P^+(\mu_k, x)\left(\frac{dv_k}{dx} - i\mu_k\Lambda v_k\right) = 0, \quad 1 \le k \le N,$$

which implies

$$\frac{dv_k}{dx} = i\mu_k\Lambda v_k, \quad 1 \le k \le N.$$

The time dependence of v_k:

$$\frac{dv_k}{dt} = -\frac{ih}{\mu_k}\Lambda v_k, \quad 1 \le k \le N,$$

can be determined similarly through an associated RH problem with the variable t. Summing up, we obtain (4.2). Finally, from (4.5), we get

$$P_1^+ = -\sum_{k,l=1}^{N} v_k(M^{-1})_{kl}\widehat{v}_l,$$

and thus by (4.4), the N-soliton solution to the system of (3.4) is (4.1). □

5. Zakharov–Shabat System with Real and Anti-symmetric Potential

As is shown in third chapter that, in the cases of real and anti-symmetric potential $-p = q = r$ the systems (3.4) becomes the (3.1), (3.2) which compatibility condition of these systems is the CKG equation.

Theorem 2. *In the case of real and anti-symmetric potential $-p = q = r$, if $h = \frac{1}{4}$ and $Q \to 0$ as $|x| \to \infty$ in the systems (3.4) then the following statements hold:*

a) The s_{11} and s_{22} have equal number zeros $\{\mu_k \in \mathbb{C}^+,\ 1 \le k \le N\}$ on upper half plane and $\{\widehat{\mu}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$ lower half plane, respectively and $\widehat{\mu}_k = -\mu_k$.

b) The N-soliton solution to the system of CKG is

$$r = 2i\sum_{k=1}^{N} v_{k,1}(M^{-1})_{kk}v_{k,2}, \tag{5.1}$$

where $v_k = (v_{k,1}, v_{k,2})^T$, $1 \le k \le N$ with

$$v_k(x,t) = e^{i\mu_k\Lambda x - \frac{i}{4\mu_k}\Lambda t}v_{k,0}, \quad 1 \le k \le N, \tag{5.2}$$

where $v_{k,0}$ $1 \le k \le N$ are arbitrary constant column vector and $M = (M_{kl})_{N\times N}$ is a square matrix whose entries are

$$M_{kk} = \frac{[v_k]^T v_k}{2\mu_k}, \quad 1 \le k \le N. \tag{5.3}$$

Proof. Let the eigenfunctions Φ, Ψ, $\widehat{\Phi}$, and $\widehat{\Psi}$ be defined with the following boundary conditions for the eigenvalue μ in ZS system

$$\Phi \sim \begin{pmatrix}1\\0\end{pmatrix}e^{-i\mu x}, \quad \Psi \sim \begin{pmatrix}0\\1\end{pmatrix}e^{i\mu x},$$
$$x \to -\infty \qquad\qquad x \to +\infty$$
$$\widehat{\Phi} \sim \begin{pmatrix}0\\-1\end{pmatrix}e^{i\mu x}, \quad \widehat{\Psi} \sim \begin{pmatrix}1\\0\end{pmatrix}e^{-i\mu x}.$$
$$x \to -\infty \qquad\qquad x \to +\infty$$

For these eigenfunctions, $W(\Phi, \widehat{\Phi}) = -1$ and $W(\Psi, \widehat{\Psi}) = -1$, where W is the Wronskian. Therefore, the eigenfunctions Ψ and $\widehat{\Psi}$ are linearly independent. Hence the functions Φ and $\widehat{\Phi}$ can be written as

$$\Phi = s_{11}(\mu)\widehat{\Psi} + s_{12}(\mu)\Psi,$$

$$\widehat{\Phi} = s_{22}(\mu)\Psi + s_{21}(\mu)\widehat{\Psi}.$$

The scattering matrix is usually defined as

$$S = \begin{pmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix}.$$

Using $W(\Phi, \overline{\Phi}) = -1$ equation,

$$-s_{11}(\mu)s_{22}(\mu) + s_{12}(\mu)s_{21}(\mu) = -1,$$

is obtained. The functions $e^{i\mu x}\Phi$, $e^{-i\mu x}\Psi$ admits analytical continuation into upper half-plane of μ and $e^{-i\mu x}\widehat{\Phi}$, $e^{i\mu x}\widehat{\Psi}$ admits analytical continuation into lower half-plane of μ. It follows from $s_{11}(\mu) = W(\Phi, \Psi)$ is analytic in the upper half-plane and $s_{22}(\mu) = -W(\widehat{\Phi}, \widehat{\Psi})$ is analytic in the lower half-plane; moreover, they tends to unity as $|\mu| \to \infty$.

The function $s_{11}(\mu)$ has a zero in the upper half plane and $s_{22}(\mu)$ has a zero in the lower half plane. If the zeros of $s_{11}(\mu)$ are called μ_k, $k = 1, 2, \dots, N$ then at $\mu = \mu_k$, Φ and Ψ proportional such that

$$\Phi = c_k\Psi.$$

Similarly, if the zeros of $s_{22}(\mu)$ are called $\widehat{\mu}_k$, $k = 1, 2, \dots, \widehat{N}$ then at $\mu = \widehat{\mu}_k$, $\widehat{\Phi}$ and $\widehat{\Psi}$ proportional such that

$$\widehat{\Phi} = \widehat{c_k}\widehat{\Psi}.$$

In this case, s_{12} and s_{21} can be expanded to $c_k = s_{12}(\mu_k)$ and $\widehat{c_k} = s_{21}(\widehat{\mu}_k)$. In this case, $s_{11}(\mu)$ and $s_{22}(\mu)$ are analytic on the real axis, and are also analytic in the upper half plane and the lower half plane. This means that $s_{11}(\mu)$ has only a finite number of zeros for $\mathrm{Im}(\mu) \geqq 0$.

The special type of relationship between p and q, i.e., $p = -q$, a case of special interest. If, in ZS system, we put

$$p = -q$$

then involution arises in the solutions of ZS system. In other words, if $\Phi = \binom{\Phi_1}{\Phi_2}$ and $\Psi = \binom{\Psi_1}{\Psi_2}$ are the solutions of ZS system with real μ then the columns

$$\widehat{\Phi}(x,\mu) = \begin{pmatrix}\Phi_2(x,-\mu)\\ -\Phi_1(x,-\mu)\end{pmatrix} \quad\text{and}\quad \widehat{\Psi}(x,\mu) = \begin{pmatrix}\Psi_2(x,-\mu)\\ -\Psi_1(x,-\mu)\end{pmatrix}$$

also are solutions and implies that symmetry relations

$$-s_{22}(\mu) = s_{11}(-\mu), \quad s_{21}(\mu) = s_{12}(-\mu)$$

and

$$\widehat{N} = N, \quad \widehat{\mu}_k = -\mu_k, \quad \widehat{c_k} = c_k.$$

Because $\widehat{N} = N$, the s_{11} and s_{22} have equal number zeros $\{\mu_k \in \mathbb{C}^+,\ 1 \le k \le N\}$ on upper half plane and $\{\widehat{\mu}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$ lower half plane, respectively and $\widehat{\mu}_k = -\mu_k$.

If we take into account

$$\mu_k = -\widehat{\mu}_k$$

in (4.1) by suitable choosing of the arbitrary column $v_{k,0}$ and row $\widehat{v}_{k,0}$, $1 \le k \le N$ constant vectors, for example, in the case $M_{kl} = 0$, $1 \le k, l \le N$; $k \ne l$, and $\widehat{v}_{k,0} = [v_{k;0}]^T$, (4.1) becomes

$$-p = q = 2i\sum_{k=1}^{N} v_{k,1}(M^{-1})_{kk}v_{k,2}.$$

□

Example 1. Single soliton solution of CKG equation: Let $N = 1$ in formulas (5.1) and (5.2). Then

$$r = 2iv_{1,1}(M^{-1})_{11}v_{1,2} \tag{5.4}$$

and

$$\begin{aligned} v_{1,1} &= e^{-i\mu_1 x + \frac{i}{4\mu_1}t}\omega_1 + \omega_2,\\ v_{1,2} &= \omega_1 + e^{i\mu_1 x - \frac{i}{4\mu_1}t}\omega_2, \end{aligned} \tag{5.5}$$

are obtained, where ω_1 and ω_2 are constants.

The formula (5.3) becomes

$$(M^{-1})_{11} = \frac{2\mu_1}{(e^{-i\mu_1 x + \frac{i}{4\mu_1}t}\omega_1 + \omega_2)^2 + (\omega_1 + e^{i\mu_1 x - \frac{i}{4\mu_1}t}\omega_2)^2} \tag{5.6}$$

since $N = 1$. If we take (5.5) and (5.6) into account (5.4) we obtain

$$r = \frac{4i\mu_1 e^{i\mu_1 x - \frac{i}{4\mu_1}t}}{1 + e^{2i\mu_1 x - \frac{i}{2\mu_1}t}}. \tag{5.7}$$

The case $N = 1$ means that $s_{11}(\mu)$ have only one zero on the imaginary axis. Let $s_{11}(\mu)$ have zero at the point $\mu_1 = i\kappa_1$ where $\kappa_1 > 0$. Then (5.7) or

$$r(x,t) = -\frac{4\kappa_1 e^{\kappa_1 x + \frac{t}{4\kappa_1}}}{1 + e^{2\kappa_1 x + \frac{t}{2\kappa_1}}}$$

presents a single soliton moving a velocity $\frac{1}{4\kappa_1^2}$.

6. Conclusion

In this paper, we study the soliton solutions of the CKG equation coupled with a scalar field which shares the same bilinear form with the sine-Gordon equation. We found a Lax pair of the CKG, of the negative order AKNS type. The spectral problem is the ZS system with real and anti-symmetric potential. This makes possible to use the inverse scattering method's technique via RH problem for obtaining and analyzing the soliton solutions of the CKG. This method provides us to show the complete integrability of the CKG equation. On the other side the various extensions and generalizations of the inverse scattering method via RH problem have been discovered, it seems that many different integrable Klein–Gordon type equations still remains to be found.

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