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On the solution of the operator Riccati equations and invariant subspaces in the weighted Bergman space of the unit ball

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Abstract. We consider the Riccati operator equations on the weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ of the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ and investigate the properties of their solutions. Our discussion uses the Berezin symbols method.

1. Introduction and background

Through the paper, $\mathbb{B}_n$ will denote the unit ball in $\mathbb{C}^n$. Let v be the normalized Lebesgue volume measure on $\mathbb{B}_n$. For $-1 < \alpha < +\infty$, we denote by v_α the measure on $\mathbb{B}_n$ defined by

$$dv_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha dv(z),$$

where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant such that $v_\alpha(\mathbb{B}_n) = 1$. For $1 \le p < \infty$, we write $\|.\|_{\alpha,p}$ for the norm in $L^p(\mathbb{B}_n, dv_\alpha)$ and $\langle ., . \rangle_\alpha$ for the inner product on $L^2(\mathbb{B}_n, dv_\alpha)$. Recall that the Bergman space $A_\alpha^2(\mathbb{B}_n)$ is the space of holomorphic functions which are square-integrable with respect to measure dv_α on $\mathbb{B}_n$. It is known that (see, for instance, Hedenmalm, Korenblum and Zhu [17]) the reproducing kernel K_w^α and normalized reproducing kernel k_w^α of the space $A_\alpha^2(\mathbb{B}_n)$ are given by

$$K_w^\alpha(z) := \frac{1}{(1 - \langle z, w \rangle)^{n+\alpha+1}}$$

and

$$k_w^\alpha(z) := \frac{\left(1 - |w|^2\right)^{\frac{n+\alpha+1}{2}}}{(1 - \langle z, w \rangle)^{n+\alpha+1}},$$

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respectively, for $z, w \in \mathbb{B}_n$. The reproducing property means that

$$\langle f, K_w^\alpha \rangle_\alpha = f(w)$$

for every $f \in A_\alpha^2(\mathbb{B}_n)$ and all $z, w \in \mathbb{B}_n$. The orthogonal projection P_α of $L^2(\mathbb{B}_n, dv_\alpha)$ onto $A_\alpha^2(\mathbb{B}_n)$ is given by

$$(P_\alpha f)(w) := \langle f, K_w^\alpha \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \frac{1}{(1 - \langle w, z \rangle)^{n+\alpha+1}} dv_\alpha(z),$$

for $f \in L^2(\mathbb{B}_n, dv_\alpha)$ and $w \in \mathbb{B}_n$. Given $h \in L^1(\mathbb{B}_n, dv_\alpha)$, the Toeplitz operator

$$T_h : A_\alpha^2(\mathbb{B}_n) \to A_\alpha^2(\mathbb{B}_n)$$

and multiplication operator

$$M_h : A_\alpha^2(\mathbb{B}_n) \to L^2(\mathbb{B}_n, dv_\alpha)$$

are defined, respectively, by

$$(T_h f)(z) = \int_{\mathbb{B}_n} \frac{f(w) h(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} dv_\alpha(w) \tag{1}$$

and

$$M_h(f) = hf.$$

Recall that for a bounded linear operator A on Bergman space $A_\alpha^2(\mathbb{B}_n)$, its Berezin symbol (transform) is defined by (see [2–5, 9, 15, 17])

$$\widetilde{A}(z) := \langle Ak_z(w), k_z(w) \rangle, \ z, w \in \mathbb{B}_n.$$

Since by the Cauchy-Schwarz inequality

$$\left| \widetilde{A}(z) \right| \leq \|Ak_z(w)\| \, \|k_z(w)\| \leq \|A\|$$

for all $z \in \mathbb{B}_n$, the Berezin symbol $\widetilde{A}$ is a bounded complex-valued function on $\mathbb{B}_n$. The present paper is motivated mostly by the papers [10, 16]. Namely, in the present article, by using the Berezin symbol technique, we study the operator Riccati equation

$$XAX + XB - CX - D = 0 \tag{2}$$

with coefficient operators A, B, C, D in $\mathcal{B}\left(A_\alpha^2(\mathbb{B}_n)\right)$.

We also investigate invariant subspaces of a Toeplitz operator on the weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ in terms of Berezin symbols.

Note that Berezin symbol of operators plays important role in operator theory (see [5, 9, 17]). In particular, it is important in the characterization of compact operators (see Nordgren and Rosenthal [15]), including compact Toeplitz operators (see [1, 17], and references therein). The method of Berezin symbols is motivated by its connection with quantum physics, see Berezin [2, 3].

2. Riccati equation on the set of Toeplitz operators

For $f \in L^1(\mathbb{B}_n, dv_\alpha)$, we define the Berezin transform (symbol) of f to be the function $\widetilde{f}$, i.e.,

$$\widetilde{f}(z) := \int_{\mathbb{B}_n} f(w) \left|k_z^\alpha(w)\right|^2 dv_\alpha(w). \tag{3}$$

If f is bounded, then $\widetilde{f}$ is a bounded function on $\mathbb{B}_n$.

According to the definition of Berezin transform, the mean oscillation of f in the weighted Bergman metric is the function $MO(f)(z)$ defined on $\mathbb{B}_n$ by (see, [6])

$$MO(f)(z) := \widetilde{\left|f\right|^2}(z) - \left|\widetilde{f}(z)\right|^2.$$

For $z \in \mathbb{B}_n$, let φ_z be the automorphism of $\mathbb{B}_n$ such that $\varphi_z(0) = z$ and $\varphi_z = \varphi_z^{-1}$. Thus, we have the change-of-variable formula

$$\int_{\mathbb{B}_n} h(\varphi_z(w)) \left|k_z^\alpha(w)\right|^2 dv_\alpha(w) = \int_{\mathbb{B}_n} h(w)\, dv_\alpha(w)$$

for every $h \in L^1(\mathbb{B}_n, dv_\alpha)$.

For $f \in L^1(\mathbb{B}_n, dv_\alpha)$, the Berezin transform of Toeplitz operator T_f coincides the Berezin transform of the function f, that is $\widetilde{T_f}(z) = \widetilde{f}(z)$ for all $z \in \mathbb{B}_n$. For more properties and facts of the Berezin transform, see, for example, Engliš [5], Nordgren and Rosenthal [15] and Zhu [17].

Recall that $BMO_\partial := BMO_\partial(\mathbb{B}_n)$ is defined to be the space of functions f such that

$$\left\|f\right\|_{BMO_\partial} := \sup_{z\in\mathbb{B}_n} MO(f)(z) < +\infty.$$

Let $VMO_\partial := VMO_\partial(\mathbb{B}_n)$ be the subspace of BMO_∂ consisting of functions f with

$$\lim_{z\to\partial\mathbb{B}_n} \left(\widetilde{\left|f\right|^2}(z) - \left|\widetilde{f}(z)\right|^2\right) = 0.$$

Also, we denote by $BA := BA(\mathbb{B}_n)$ (see [14]) the space of functions f with

$$\left\|f\right\|_{BA} := \sup_{z\in\mathbb{B}_n} \left[\widetilde{\left|f\right|^2}(z)\right]^{1/2},$$

and finally, we define $VA := VA(\mathbb{B}_n)$ (see [14]) as the subspace of BA consisting of functions f satisfying

$$\lim_{z\to\partial\mathbb{B}_n} \widetilde{\left|f\right|^2}(z) = 0.$$

It is easy to see that if $f \in VA$ then $f \in VMO_\partial$. For each $f \in VA$, it can be obtained by Hölder inequality that

$$\left|\widetilde{f(z)}\right| = \int_{\mathbb{B}_n} \left|f(w)\right| |k_z(w)|^2 dv_\alpha(w) \le \left[\widetilde{\left|f\right|^2}(z)\right]^{1/2} < +\infty.$$

Recall also that $H^\infty = H^\infty(\mathbb{B}_n)$ is the space of holomorphic functions f on $\mathbb{B}_n$ such that

$$\left\|f\right\|_{H^\infty} := \sup_{z\in\mathbb{B}_n} \left|f(z)\right| < +\infty.$$

For a function $\varphi \in BMO_\partial(\mathbb{B}_n)$, we define the Toeplitz operator T_φ with symbol φ on $A^2_\alpha(\mathbb{B}_n)$ by

$$T_\varphi f = P_\alpha(\varphi f) = \int\limits_{\mathbb{B}_n} \frac{f(w)\varphi(w)}{(1-\langle z,w\rangle)^{n+\alpha+1}} dv_\alpha(w) \text{ (see (1))}$$

for all $f \in A^2_\alpha(\mathbb{B}_n)$. For more information about Toeplitz operators, the reader can find, for example, in [1, 5, 7, 12, 13, 17].

For $\varphi \in H^\infty$, T_φ is just the operator of multiplication by φ on the space $A^2_\alpha(\mathbb{B}_n)$, i.e., $T_\varphi f = \varphi f$, $f \in A^2_\alpha(\mathbb{B}_n)$. Then we have that

$$\begin{aligned}\left\langle T^*_\varphi k^\alpha_z, f\right\rangle &= \left\langle k^\alpha_z, T_\varphi f\right\rangle = \langle k^\alpha_z, \varphi f\rangle \\ &= \overline{\langle \varphi f, k^\alpha_z\rangle} = \overline{(\varphi f)(z)} = \left\langle \overline{\varphi(z)} k^\alpha_z, f\right\rangle,\end{aligned}$$

hence, $T^*_\varphi k^\alpha_z = \overline{\varphi(z)} k^\alpha_z$ for all $z \in \mathbb{B}_n$.

In this section, we study Riccati operator equation (2) on the set of Toeplitz operators $T_\varphi \in \mathcal{B}\left(A^2_\alpha(\mathbb{B}_n)\right)$ in terms of Berezin symbols, which firstly studied by Karaev [10] in the Hardy Hilbert space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D}$ of $\mathbb{C}$. In the sequel, this approach attracted attention of several authors in [7, 8, 11, 12, 14]. The following extension of Engliš's result, [5], is essential in what follows, so, we start with the proof of the following auxiliary lemma.

Lemma 2.1. *Let $\varphi \in VMO_\partial(\mathbb{B}_n)$. Then*

$$\lim_{z\to\partial\mathbb{B}_n} \left\| T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^\alpha_z \right\|_{A^2_\alpha(\mathbb{B}_n)} = 0.$$

Proof. Indeed, it is easy to see that

$$\begin{aligned}\left\| T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^\alpha_z \right\|^2_{A^2_\alpha(\mathbb{B}_n)} &= \left\langle T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^\alpha_z, T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^z_\alpha \right\rangle \\ &= \left\| T_\varphi k^\alpha_z \right\|^2 - \overline{\widetilde{\varphi}(z)} \left\langle T_\varphi k^\alpha_z, k^\alpha_z\right\rangle - \widetilde{\varphi}(z)\left\langle k^\alpha_z, T_\varphi k^\alpha_z\right\rangle + \left|\widetilde{\varphi}(z)\right|^2.\end{aligned}$$

Since $\widetilde{T_\varphi}(z) = \widetilde{\varphi}(z)$, we have

$$\left\| T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^\alpha_z \right\|^2_{A^2_\alpha(\mathbb{B}_n)} = \left\| T_\varphi k^\alpha_z \right\|^2 - \left|\widetilde{\varphi}(z)\right|^2.$$

On the other hand,

$$\left\| T_\varphi k^\alpha_z \right\|^2 = \left\| P_\alpha(\varphi k^\alpha_z)\right\|^2 \le \left\| \varphi k^\alpha_z \right\|^2_{L^2(\mathbb{B}_n, dV_\alpha)} = \widetilde{\left|\varphi\right|^2}(z).$$

Then

$$\left\| T_\varphi k^\alpha_z - \widetilde{\varphi}(z) k^\alpha_z \right\|^2_{A^2_\alpha(\mathbb{B}_n)} \le \widetilde{\left|\varphi\right|^2}(z) - \left|\widetilde{\varphi}(z)\right|^2.$$

For $\varphi \in VMO_\partial$, we have

$$\widetilde{\left|\varphi\right|^2}(z) - \left|\widetilde{\varphi}(z)\right|^2 \to 0,\ z \to \partial\mathbb{B}_n \text{ non-tangentially,}$$

which proves the lemma. □

In next theorem, we consider the case that Toeplitz operator is a solution of the Riccati equation (2) on the space $A^2_\alpha(\mathbb{B}_n)$.

Theorem 2.2. *Suppose that $B = T_\varphi^*$, $C = T_\psi$ such that φ, ψ are nonconstant functions on H^∞, and let A and D be bounded linear operators on $A_\alpha^2(\mathbb{B}_n)$. If $h \in VA(\mathbb{B}_n)$ and the corresponding Toeplitz operator T_h is a solution of the operator Riccati equation (2), then*

$$\lim_{z\to\partial\mathbb{B}_n}\left(\widetilde{A}(z)\widetilde{h}^2(z) + \left(\overline{\varphi(z)} - \psi(z)\right)\widetilde{h}(z) - \widetilde{D}(z)\right) = 0. \tag{4}$$

Proof. In fact, since T_h is a solution of equation (2), then

$$T_hAT_h + T_hT_\varphi^* - T_\psi T_h - D = 0.$$

Then, $\left[T_hAT_h + T_hT_\varphi^* - T_\psi T_h - D\right]^{\sim}(z) = 0$ for all $z \in \mathbb{B}_n$, that is

$$\left\langle\left(T_hAT_h + T_hT_\varphi^* - T_\psi T_h - D\right)k_z^\alpha, k_z^\alpha\right\rangle = 0,\ \forall z \in \mathbb{B}_n.$$

Therefore

$$\begin{aligned}
0 &= \langle T_hAT_hk_z^\alpha, k_z^\alpha\rangle + \left\langle T_hT_\varphi^*k_z^\alpha, k_z^\alpha\right\rangle - \left\langle T_\psi T_hk_z^\alpha, k_z^\alpha\right\rangle - \langle Dk_z^\alpha, k_z^\alpha\rangle \\
&= \left\langle AT_hk_z^\alpha, T_h^*k_z^\alpha - \widetilde{T}_h^*(\lambda)k_z^\alpha\right\rangle + \widetilde{T}_h(\lambda)\langle AT_hk_z^\alpha, k_z^\alpha\rangle \\
&+ \left\langle T_h\left(T_h^*k_z^\alpha - \widetilde{T_\varphi^*}(\lambda)k_z^\alpha\right), k_z^\alpha\right\rangle + \widetilde{T_\varphi^*}(z)\widetilde{T}_h(z) \\
&- \left\langle T_\psi\left(T_hk_z^\alpha - \widetilde{T}_h(\lambda)k_z^\alpha\right), k_z^\alpha\right\rangle - \widetilde{T}_\psi(\lambda)\widetilde{T}_h(z) - \widetilde{D}(z) \\
&= \left\langle AT_hk_z^\alpha, T_h^*k_z^\alpha - \widetilde{T}_h^*(\lambda)k_z^\alpha\right\rangle \\
&+ \widetilde{T}_h(\lambda)\left[\left\langle A\left(T_hk_z^\alpha - \widetilde{T}_h(\lambda)k_z^\alpha\right), k_z^\alpha\right\rangle + \widetilde{T}_h(\lambda)\widetilde{A}(z)\right] \\
&+ \left\langle T_\varphi^*k_z^\alpha - \widetilde{T_\varphi^*}(\lambda)k_z^\alpha, T_h^*k_z^\alpha\right\rangle + \widetilde{T_\varphi^*}(\lambda)\widetilde{T}_h(z) - \left\langle T_hk_z^\alpha - \widetilde{T}_h(\lambda)k_z^\alpha, T_\psi^*k_z^\alpha\right\rangle \\
&- \widetilde{T}_\psi(\lambda)\widetilde{T}_h(z) - \widetilde{D}(z) \\
&= \left[\widetilde{A}(z)\widetilde{h}^2(z) + \left(\overline{\varphi(z)} - \psi(z)\right)\widetilde{h}(z) - \widetilde{D}(z)\right] + \left\langle AT_hk_z^\alpha, T_h^*k_z^\alpha - \overline{\widetilde{h}}(z)k_z^\alpha\right\rangle \\
&+ \widetilde{h}(z)\left\langle T_hk_z^\alpha - \widetilde{h}(z)k_z^\alpha, A^*k_z^\alpha\right\rangle + \left\langle\overline{\varphi(z)}k_z^\alpha - \overline{\psi(z)}k_z^\alpha, T_h^*k_z^\alpha\right\rangle,
\end{aligned}$$

which by Lemma 2.1 yields

$$\begin{aligned}
\left|\widetilde{A}(z)\widetilde{h}^2(z) + \left(\overline{\varphi(z)} - \psi(z)\right)\widetilde{h}(z) - \widetilde{D}(z)\right| &\le \|AT_h\|\left\|T_h^*k_z^\alpha - \overline{\widetilde{h}}(z)k_z^\alpha\right\| \\
&+ \|T_h\|\,\|A\|\left\|T_hk_z^\alpha - \widetilde{h}(z)k_z^\alpha\right\| \to 0 \text{ as } z \to \partial\mathbb{B}_n.
\end{aligned}$$

This implies the desired assertion (4), which proves the theorem. □

Corollary 2.3. *Suppose that the limits $\widetilde{A}(\xi) := \lim_{z\to\partial\mathbb{B}_n}\widetilde{A}(z)$ and $\widetilde{D}(\xi) := \lim_{z\to\partial\mathbb{B}_n}\widetilde{D}(z)$ exist for almost all points $\xi \in \partial\mathbb{B}_n$, and verify*

$$\left(\overline{\varphi}(\xi) - \psi(\xi)\right)^2 + 4\widetilde{A}(\xi)\widetilde{D}(\xi) = 0 \tag{5}$$

for almost all points $\xi \in \partial\mathbb{B}_n$. Let $h \in VA(\mathbb{B}_n)$. If T_h is a solution of Riccati equation (2), then

$$\lim_{z\to\xi\in\partial\mathbb{B}_n}\widetilde{h}^2(z) = -\frac{\widetilde{D}(\xi)}{\widetilde{A}(\xi)}$$

for almost all $\xi \in \partial\mathbb{B}_n$.

Proof. Indeed, since T_h satisfies equation (2), we have obtain by Theorem 2.2 that

$$\lim_{z\to\partial\mathbb{B}_n}\left(\widetilde{A}(z)\widetilde{h}^2(z)+\left(\overline{\varphi(z)}-\psi(z)\right)\widetilde{h}(z)-\widetilde{D}(z)\right)=0.$$

Further,

$$\begin{aligned}
&\lim_{z\to\partial\mathbb{B}_n}\widetilde{A}(z)\left[\widetilde{h}^2(z)+\frac{\overline{\varphi(z)}-\psi(z)}{\widetilde{A}(z)}\widetilde{h}(z)-\frac{\widetilde{A}(z)\widetilde{D}(z)}{\widetilde{A}^2(z)}\right]\\
&=\lim_{z\to\partial\mathbb{B}_n}\widetilde{A}(z)\left[\widetilde{h}^2(z)+\frac{\overline{\varphi(z)}-\psi(z)}{\widetilde{A}(z)}\widetilde{h}(z)+\frac{\left(\overline{\varphi(z)}-\psi(z)\right)^2}{4\widetilde{A}^2(z)}\right]\\
&=\lim_{z\to\partial\mathbb{B}_n}\widetilde{A}(z)\left(\widetilde{h}(z)+\frac{\overline{\varphi(z)}-\psi(z)}{2\widetilde{A}(z)}\right)^2=0.
\end{aligned}$$

In particular,

$$\lim_{z\to\xi\in\partial\mathbb{B}_n}\widetilde{h}(z)=\frac{\psi(\xi)-\overline{\varphi(\xi)}}{2\widetilde{A}(\xi)},$$

hence we have that

$$\lim_{z\to\xi\in\partial\mathbb{B}_n}\widetilde{h}^2(z)=-\frac{\widetilde{D}(\xi)}{\widetilde{A}(\xi)}$$

for almost all $\xi\in\partial\mathbb{B}_n$, as desired. □

3. Invariant subspaces of compact Toeplitz operators on $A^2_\alpha(\mathbb{B}_n)$

Note that the solvability of the Riccati operator equation in concrete operator classes is an important problem of operator theory in the Hilbert space H. For example, the existence of a nontrivial solution of equation (2) for fixed $A\in\mathcal{B}(H)$, $B=D=0$ and $C=A$ on the set $\mathcal{P}_h$ of all orthogonal projections $P\in\mathcal{B}(H)$ is equivalent to the positive solution of the famous Invariant Subspace Problem in the infinite dimensional separable complex Hilbert space, since $TE\subset E$ if and only if

$$(I-P_E)\,TP_E=0, \tag{6}$$

where $P_E:H\to E$ is an orthogonal projection onto the closed subspace $E\subset H$. In this section, we discuss the structure of invariant subspaces of Toeplitz operator T_f on the weighted Bergman space $A^2_\alpha(\mathbb{B}_n)$ in terms of Berezin symbols.

For any $p\geq 1$, VMO^p_α denotes the subspace BMO^p_α consisting of functions f such that

$$\lim_{z\to\partial\mathbb{B}_n}\left\|f\circ\varphi_z-\widetilde{f}(z)\right\|_{\alpha,p}=0.$$

The following theorem is contained in [6].

Theorem 3.1. *([6]) Suppose $f\in VMO^1_\alpha$ and $\left|f\right|/(1-|z|)^{4(n+\alpha+1)}$ is bounded on the unit ball $\mathbb{B}_n$. Then T_f is compact operator on $A^2_\alpha(\mathbb{B}_n)$ if and only if $\widetilde{f}(z)\to 0$ as $z\to\partial\mathbb{B}_n$.*

Our next result is the following.

Theorem 3.2. *Suppose $f \in VMO^1_\alpha$, $\left|f\right| / (1-|z|)^{4(n+\alpha+1)}$ is bounded on $\mathbb{B}_n$ and $\widetilde{f}(z) \to 0$ as $z \to \partial\mathbb{B}_n$. Let $E \subset A^2_\alpha(\mathbb{B}_n)$ be a closed subspace of $A^2_\alpha(\mathbb{B}_n)$. If $T_\varphi E \subset E$, i.e., E is invariant under the Toeplitz operator T_f, then*

$$\left|\left(T_f K^{\alpha,E}_z\right)(z) - \widetilde{f}(z) K^{\alpha,E}_z(z)\right| = o\left(\left(1-|z|^2\right)^{-(n+\alpha+1)}\right) \text{ as } z \to \partial\mathbb{B}_n, \tag{7}$$

where $K^{\alpha,E}_z$ denotes the reproducing kernel of the subspace E, which is equal $P_E K^\alpha_z$.

Proof. Suppose $T_f E \subset E$. Then it means that $(I - P_E) T_f P_E = 0$, and hence $P_E T_f P_E - T_f P_E = 0$, that is P_E is a solution of equation (6) where $T = T_f$. It is then clear from the above equality that

$$\widetilde{T_f P_E}(z) - \widetilde{P_E T_f P_E}(z) = 0$$

for all $z \in \mathbb{B}_n$. So, we have that

$$\begin{aligned} 0 &= \left\langle P_E k^\alpha_z, T^*_f k^\alpha_z \right\rangle - \left\langle T_f P_E k^\alpha_z, P_E k^\alpha_z \right\rangle \\ &= \left\langle P_E k^\alpha_z, T_{\overline{f}} k^\alpha_z - \widetilde{\overline{f}}(z) k^\alpha_z \right\rangle + \frac{\widetilde{f}(z)}{K^\alpha_z(z)} \left\langle K^{\alpha,E}_z, K^\alpha_z \right\rangle - \frac{1}{K^\alpha_z(z)} \left\langle T_f K^{\alpha,E}_z, K^{\alpha,E}_z \right\rangle. \end{aligned}$$

By considering that $T_f E \subset E$, and therefore $T_f K^{\alpha,E}_z \in E$, we have that

$$\left\langle T_f K^{\alpha,E}_z, K^{\alpha,E}_z \right\rangle = \left(T_f K^{\alpha,E}_z\right)(z)$$

for all $z \in \mathbb{B}_n$. On the other hand, we obtain from Theorem 3.1 and the Cauchy-Schwarz inequality that

$$\begin{aligned} &\lim_{z\to\xi\in\partial\mathbb{B}_n} \frac{1}{K^\alpha_z(z)} \left|\left(T_f K^{\alpha,E}_z\right)(z) - \widetilde{f}(z) K^{\alpha,E}_z(z)\right| \\ &= \lim_{z\to\xi\in\partial\mathbb{B}_n} \left(1-|z|^2\right)^{(n+\alpha+1)} \left|\left(T_f K^{\alpha,E}_z\right)(z) - \widetilde{f}(z) K^{\alpha,E}_z(z)\right| \\ &= \lim_{z\to\xi\in\partial\mathbb{B}_n} \left|\left\langle P_E k^\alpha_z, T_{\overline{f}} k^\alpha_z - \widetilde{\overline{f}}(z) k^\alpha_z \right\rangle\right| \\ &\leq \lim_{z\to\xi\in\partial\mathbb{B}_n} \left\|P_E k^\alpha_z\right\| \left\|T_{\overline{f}} k^\alpha_z - \widetilde{\overline{f}}(z) k^\alpha_z\right\| \\ &\leq \lim_{z\to\xi\in\partial\mathbb{B}_n} \left\langle T_{\overline{f}} k^\alpha_z - \widetilde{\overline{f}}(z) k^\alpha_z, T_{\overline{f}} k^\alpha_z - \widetilde{\overline{f}}(z) k^\alpha_z \right\rangle^{\frac{1}{2}} \\ &= \lim_{z\to\xi\in\partial\mathbb{B}_n} \left(\left\|T_{\overline{f}} k^\alpha_z\right\|^2 - \left|\widetilde{f}(z)\right|^2\right)^{1/2} \text{ (since } T k^\alpha_z - \widetilde{T}(z) k^\alpha_z \perp k^\alpha_z \text{ for any } T \in B\left(A^2_\alpha(\mathbb{B}_n)\right)\text{)}. \end{aligned}$$

Since $\lim_{z\to\partial\mathbb{B}_n} \widetilde{f}(z) = 0$, by Theorem 3.1, T_f is compact on $A^2_\alpha(\mathbb{B}_n)$. On the other hand, the normalized reproducing kernel k^α_z converges weakly to 0 as z tends $\partial\mathbb{B}_n$, and therefore $\left\|T_{\overline{f}} k^\alpha_z\right\| \to 0$ as $z \to \partial\mathbb{B}_n$. Thus, we deduce from the latter inequality that

$$\lim_{z\to\partial\mathbb{B}_n} \left(1-|z|^2\right)^{(n+\alpha+1)} \left|\left(T_f K^{\alpha,E}_z\right)(z) - \widetilde{f}(z) K^{\alpha,E}_z(z)\right| = 0,$$

which gives (7). The proof is completed. □

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