Nazim B. KERIMOV, Elchin J. IBADOV

ON NECESSARY CONDITIONS OF BASICITY OF A SYSTEM OF EIGEN-FUNCTIONS OF SECOND ORDER DISCONTINUOUS OPERATORS

Abstract

In the paper the basicity properties of a system of eigen-functions of the differential operator $Lu = u'' + q(x)\,u$ are investigated, where $q(x)$ is a complex-valued potential from the space $L_1(0,1)$. Following V.A.Ilin we proceed from the generalized treatment of eigen-functions and they can have discontinuity of the first order. The necessary conditions of basicity of a system of eigen-functions in the space L_p $(1 < p < \infty)$ in terms of eigen-values are obtained.

1. Basic definitions. Formulation of results. Let's on interval $G = (0;1)$ of real axis consider the operator

$$Lu(x) = u''(x) + q(x)\,u(x) \tag{1.1}$$

with complex-valued potential $q(x) \in L_1(0;1)$. Following the V.A.Ilin papers (see for example [1]), we'll proceed from the generalized treatment of eigen-functions of the operator (1.1).

Assume that with the help of the points

$$0 = \xi_0 < \xi_1 < \xi_2 < ... < \xi_m < \xi_{m+1} = 1$$

the interval $(0;1)$ is divided into $m+1$ intervals $\left(\xi_{l-1};\zeta_l\right),\ l = \overline{1,m+1}$.

By $D_l\left(l = \overline{1,m+1}\right)$ we denote a class of functions, absolutely continuous together with their first derivatives on the segment $\left[\xi_{l-1};\zeta_l\right]$. Let D be a class of functions having the following properties: if $f \in D$, then for each $l = \overline{1,m+1}$ there exists a function $f_l \in D_l$ such that $f = f_l$ at $\xi_{l-1} < x < \xi_l$.

Under the eigen-function of the operator L responding to complex eigen-value λ we'll understand any function $y(x) \in D$ not equal identically to zero satisfying almost everywhere in the interval $(0;1)$ the equation

$$Ly(x) + \lambda y(x) = 0.$$

Consider an arbitrary system $\{u_n(x)\}_1^\infty$ consisting of eigen-functions of the operator L that we understand in the generalized sense. The corresponding system of eigen-values will be denoted by $\{\lambda_n\}_1^\infty$. It means that each function $u_n(x)$ belongs to D and almost everywhere in the interval $(0,1)$ satisfies the equation

$$Lu_n(x) + \lambda_n u_n(x) = 0. \tag{1.2}$$

By the symbol L^* we'll denote an operator formally conjugated to the operator L, namely $L^*v = v''(x) + \overline{q(x)}v(x)$. Everywhere in the sequel it is assumed that p is a fixed number:

$$\frac{1}{p} + \frac{1}{q} = 1,\ 1 \le p < +\infty.$$

Assume for brevity

$$G_l = \left(\xi_{l-1}, \xi_l\right), \quad (f,g)_{G_l} = \int\limits_{\xi_{l-1}}^{\xi_l} f(x)\,\overline{g(x)}dx,$$

$$(f,g) = \int\limits_0^1 f(x)\,\overline{g(x)}dx, \quad \|f\|_p = \left(\int\limits_0^1 |f(x)|^p\,dx\right)^{1/p}.$$

In the sequel parallel with eigen-values λ_n we'll use the spectral parameter $\mu_n = \sqrt{\lambda_n}$, where $\sqrt{r\exp(i\varphi)} = \sqrt{r}\exp(i\varphi/2)$, $-\frac{\pi}{2} \le \varphi < \frac{3\pi}{2}$.

Let $D^{(0)}$ be a class of functions having the following properties: if $\psi \in D^{(0)}$ then for each $l = \overline{1, m+1}$ there exists a function $\psi_l \in C\left[\xi_{l-1;}, \xi_l\right]$ such that $\psi = \psi_l$ at $\xi_{l-1} < x < \xi_l$. It is obvious that any function from this class can have the discontinuity (only the first order discontinuity) only at the points ξ_l $\left(l = \overline{0, m+1}\right)$.

The basic results of the present paper are the following assertions.

Theorem 1. *Let $\{u_n(x)\}_1^\infty$ be an arbitrary system consisting of the eigen-functions of operator (1.1). If for arbitrary $f(x) \in D^{(0)}$*

$$\lim_{n\to\infty} (u_n, f)\,\|u_n\|_p^{-1} = 0, \tag{1.3}$$

then the sequence $\{\mu_n\}_1^\infty$ hasn't finite points of concentration.

Corollary 1. *Let $\{u_n(x)\}_1^\infty$ be an arbitrary system consisting of eigen-functions of operator (1.1). If $\{u_n(x)\}_1^\infty$ forms the basis of the space $L_p(G)$ $(1 < p < \infty)$, then the sequence $\{\mu_k\}_1^\infty$ hasn't finite points of concentration.*

Theorem 2. *Let the following two conditions be fulfilled:*

1) $\{u_n(x)\}_1^\infty$ is an arbitrary minimal system in $L_p(0;1)$ $(1 < p < \infty)$ consisting of eigen-functions of the operator L;

2) the system $\{\upsilon_n(x)\}_1^\infty$ is biorthogonally conjugated to $\{u_n(x)\}_1^\infty$ and consists of eigen-functions of the operator L^*.

If $\{u_n(x)\}_1^\infty$ forms basis of the space $L_p(0;1)$ then there exists the constant C_0 such that for all $n \ge 1$ it holds

$$|\operatorname{Im}\mu_n| \le C_0, \tag{1.4}$$

Remark 1.1. The second condition of theorem 2 means that the function $\upsilon_n(x)$ belongs to D and almost everywhere in G satisfies the equation $L^*\upsilon_n(x) + \overline{\lambda_n}\upsilon_n(x) = 0$.

Remark 1.2. The formulated results are easily transferred on the case of the operator $Lu(x) = u''(x) + p_1(x)u' + p_2(x)u(x)$, where $p_1(x)$ is absolutely continuous on the segment $[0;1]$ and $p_2(x) \in L_1(0,1)$.

Everywhere in future under C we'll understand a positive constant not necessarily the same.

2. The proof of theorem 1. Let the assertion of theorem 1 be not true. Then there exists finite number a and subsequence $\left\{\mu_{n_k}(x)\right\}_{k=1}^\infty$ such that $\lim\limits_{k\to\infty}\mu_{n_k} = a$.

In future we'll use the following estimations of eigen-functions constructed by V.V.Tikhomirov [2]:

$$\sup_{\xi_{l-1}<x<\xi_l}\left|u_n^{(s)}(x)\right| \le C\left(1+|\mu_n|\right)^s\left(1+|\operatorname{Im}\mu_n|\right)^{1/p}\|u_n\|_{L_p\left(\xi_{l-1;},\xi_l\right)}. \tag{2.1}$$

In these estimations $n \in \mathbf{N};\ s = 0, 1$ and $l = \overline{1, m+1}$.

By virtue of determination of eigen-functions of the operator L for each $n \in N$ there exist the functions $u_{n,l}(x) \in D_l$ $\left(l = \overline{1, m+1}\right)$ such that

$$u_{n,l}(x) \equiv u_n(x) \quad \left(\xi_{l-1} < x < \xi_l\right). \tag{2.2}$$

It is obvious that the sequence $\left\{\mu_{n_k}\right\}_{k=1}^{\infty}$ is bounded. Consequently, by virtue of inequality (2.1) and relation (2.2) we have

$$\left|u_{n_k,l}^{(s)}(x)\right| \|u_{n_k}\|_{L_p\left(\xi_{l-1};\xi_l\right)}^{-1} \le C \qquad \left(\xi_{l-1} < x < \xi_l\right), \tag{2.3}$$

where $s = 0, 1$ and $l = \overline{1, m+1}$.

Further using (2.3) at $s = 1$ we obtain that at $x, y \in \left[\xi_{l-1}; \xi_l\right]$ it holds

$$|u_{n_k,l}(x) - u_{n_k,l}(y)| \, \|u_{n_k}\|_{L_p\left(\xi_{l-1};\xi_l\right)}^{-1} =$$

$$= \left|\int_y^x u'_{n_k,l}(t)\, dt\right| \|u_{n_k}\|_{L_p\left(\xi_{l-1};\xi_l\right)}^{-1} \le C\,|x-y|\,.$$

Since $\|u_{n_k}\|_{L_p\left(\xi_{l-1};\xi_l\right)} \le \|u_{n_k}\|_p$ $\left(l = \overline{1, m+1}\right)$, then it is obvious that $\left\{u_{n_k}(x)\,\|u_{n_k}\|_p^{-1}\right\}_{k=1}^{\infty}$ is a uniformly bounded equipotentionally continuous family on $\left[\xi_{l-1}; \xi_l\right]$. Consequently, there exists a subsequence $\{n_k(1)\}$ of the subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} u_{n_k(1),1}(x) \left\|u_{n_k(1)}\right\|_p^{-1} = \psi_1(x) \, (\text{uniformlyby } x \in [\xi_0; \xi_1]), \tag{2.4}$$

where $\psi_1(x)$ is a function from the class $C[\xi_0; \xi_1]$. $\left\{u_{n_k(1),2}(x) \left\|u_{n_k(1)}\right\|_p^{-1}\right\}_{k=1}^{\infty}$ is a uniformly bounded equipotentionally continuous family on $[\xi_1; \xi_2]$. Consequently, there exists a subsequence $\{n_k(2)\}_1^{\infty}$ of the sequence $\{n_k(2)\}_1^{\infty}$ such that
$\lim_{k\to\infty} u_{n_k(2),2}(x) \left\|u_{n_k(1)}\right\|_p^{-1} = \psi_2(x)$ (uniformly by $x \in [\xi_1; \xi_2]$),
where $\psi_2(x)$ is a function from the class $C[\xi_1; \xi_2]$.

Acting absolutely similarly we'll obtain: a) there exist the sequences $\{n_k(1)\}_1^{\infty}, ..., \{n_k(m+1)\}_1^{\infty}$ such that each of these sequences are subsequences of the previous one; b) there exist the functions $\psi_l(x) \in C\left[\xi_{l-1}; \xi_l\right]$ $\left(l = \overline{1, m+1}\right)$ such that
$\lim_{k\to\infty} u_{n_k(l),l}(x) \left\|u_{n_k(l)}\right\|_p^{-1} = \psi_l(x)$ (uniformly by $x \in \left[\xi_{l-1}; \xi_l\right]$).

It is obvious that at $l = \overline{1, m+1}$ it holds
$\lim_{k\to\infty} u_{n_k(m+1),l}(x) \left\|u_{n_k(m+1)}\right\|_p^{-1} = \psi_l(x)$ (uniformly by $x \in \left[\xi_{l-1}; \xi_l\right]$).

We'll determine the function $\psi(x)$ in the following way

$$\psi(x) = \begin{cases} \psi_1(x), \text{ if } \xi_0 < x < \xi_1, \\ \psi_2(x), \text{ if } \xi_1 < x < \xi_2, \\ \dots\dots\dots\dots\dots\dots \\ \psi_{m+1}(x), \text{ if } \xi_m < x < \xi_{m+1}. \end{cases} \tag{2.5}$$

It is obvious that the function $\psi(x)$ can have only discontituity of the first order at the points $\xi_l \left(l = \overline{0, m+1}\right)$ and the sequence $\left\{u_{n_k(m+1)}(x) \left\|u_{n_k(m+1)}\right\|_p^{-1}\right\}_1^\infty$ strongly converges to the function $\psi(x)$ in the space $L_p(0;1)$. Hence for the function $\psi(x)$ w have $\|\psi\|_p = 1$.

Besides,

$$\|\psi\|_2^2 = \lim_{k\to\infty} \left(u_{n_k(m+1)}, \psi\right) \left\|u_{n_k(m+1)}\right\|_p^{-1} = 0.$$

The obtained contradiction completes the proof of theorem 1.

3. The proof of theorem 2. Note that at each of the intervals $\left(\xi_{l-1}; \xi_l,\ \right)$, $l = \overline{1, m+1}$ the representation

$$\pm 2i\mu_n u_n(t) = \left(u_n'\left(\xi_{l-1}+0\right) \pm i\mu_n u_n\left(\xi_{l-1}+0\right)\right) \exp\left[i\mu_n\left(t - \xi_{l-1}\right)\right] +$$
$$+ \left(-u'\left(\xi_l - 0\right) \pm i\mu_n u_n\left(\xi_l - 0\right)\right) \exp\left[i\mu_n\left(\xi_l - t\right)\right] -$$
$$- \int_{\xi_{l-1}}^{\xi_l} q(x)\, u_n(x) \exp\left[\pm i\mu_n |x-t|\right] dt. \tag{3.1}$$

is true.

In order to prove the correctness of representation (3.1) let's multiply (3.1) by the function $\exp\left[\pm i\mu_n |x-t|\right]$, where $\xi_{l-1} < x,\ t < \xi_l$, integrate the obtained identity by x from ξ_{l-1} to ξ_l and apply the integration by parts formula.

Let the assertion of theorem 2 be not true. Then from the sequence $\{\mu_n\}_{n=1}^\infty$ we can choose such subsequence $\left\{\mu_{n_k}\right\}_{k=1}^\infty$ that

$$\lim_{k\to\infty} \left|\operatorname{Im}\mu_{n_k}\right| = \infty. \tag{3.2}$$

Assume

$$U_k(x) = u_{n_k}(x),\ V_k(x) = v_{n_k}(x),\ \Lambda_k = \mu_{n_k} \quad (k \in \mathbf{N}) \tag{3.3}$$

It is obvious that

$$\lim_{k\to\infty} \left|\operatorname{Im}\Lambda_k\right| = \infty. \tag{3.2*}$$

We'll assume (it is necessary passing to subsequence), that

$$0 < \operatorname{Im}\Lambda_k \left(\operatorname{Im}\Lambda_{k+1}\right)^{-1} \le \frac{1}{2} \quad (k \in \mathbf{N}), \tag{3.4}$$

$$\lim_{k\to\infty} \operatorname{Im}\Lambda_k \left(\operatorname{Im}\Lambda_{k+1}\right)^{-1} = 0, \tag{3.5}$$

$$\left|\operatorname{Im}\Lambda_{k+1}\right| > 1 \quad (k \in \mathbf{N}). \tag{3.6}$$

Since by the choice the all terms of the sequence $\{\operatorname{Im}\Lambda_k\}_{k=1}^\infty$ have the same signs

$$\operatorname{Im}\Lambda_k > 1 \quad (k \in \mathbf{N}). \tag{3.6*}$$

The case $\operatorname{Im}\Lambda_k < -1 \quad (k \in \mathbf{N})$ is considered absolutely analogously.

By virtue of our notation and the second condition of theorem 1 at $k \in \mathbf{N}$ we have

$$(U_k, V_k) = 1,\ (U_k, V_{k+1}) = 0. \tag{3.7}$$

In view of notation (1.3) at $x \in G_l = \left(\xi_{l-1}; \xi_l\right)$ and $k \in N$ the representation

$$U_k(x) = H_{k,l}^{(1)} \left(2\operatorname{Im}\Lambda_k\right)^{1/p} \exp\left[i\Lambda_k\left(x-\xi_{l-1}\right)\right] +$$

$$+H_{k,l}^{(2)} \left(2\operatorname{Im}\Lambda_k\right)^{1/p} \exp\left[i\Lambda_k\left(\xi_l - x\right)\right] +$$

$$+\frac{1}{2i\Lambda_k}\int_{\xi_{l-1}}^{\xi_l} q(t)\,U_k(t)\exp\left[i\Lambda_k|x-t|\right]dt, \tag{3.8}$$

$$H_{k,l}^{(1)} = \frac{U_k'\left(\xi_{l-1}+0\right) + i\Lambda_k U_k\left(\xi_{l-1}+0\right)}{2i\Lambda_k\left(2\operatorname{Im}\Lambda_k\right)^{1/p}}, \tag{3.9}$$

$$H_{k,l}^{(2)} = \frac{-U_k'\left(\xi_l - 0\right) + i\Lambda_k U_k\left(\xi_l + 0\right)}{2i\Lambda_k\left(2\operatorname{Im}\Lambda_k\right)^{1/p}} \tag{3.9$'$}$$

is true.

From estimation (2.1) it follows

$$\sup_{\xi_{l-1}<x<\xi_l}\left|U_k^{(s)}(x)\right| \le C\left|\Lambda_k\right|^s\left(\operatorname{Im}\Lambda_k\right)^{1/p}\left\|U_k\right\|_p, \tag{3.10}$$

where $k \in \mathbf{N}$ and $s = 0, 1$. Taking into account (3.9), (3.9$'$), (3.10) we conclude that at $k \in \mathbf{N}$ the inequality

$$\left|H_{k,l}^{(j)}\right| \le C\left\|U_k\right\|_p \quad (j = 1, 2) \tag{3.11}$$

is true.

Using (3.6*) and (3.10) it is easy to show that at $x \in G_l,\ k \in N$.

$$\int_{\xi_{l-1}}^{\xi_l} q(t)\,U_k(t)\exp\left[i\Lambda_k|x-t|\right]dt = \left(2\operatorname{Im}\Lambda_k\right)^{1/p}\left\|U_k\right\|_p O_1(1), \tag{3.12}$$

is true, where $O_1(1)$ is a bounded function from k, x and l.

From (3.8) and (3.12) we'll obtain

$$U_k(x) = \left(2\operatorname{Im}\Lambda_k\right)^{1/p}\left\{H_{k,l}^{(1)}\exp\left[i\Lambda_k\left(x-\xi_{l-1}\right)\right] + \right.$$

$$\left. +H_{k,l}^{(2)}\exp\left[i\Lambda_k\left(\xi_l - x\right)\right] + \frac{\left\|U_k\right\|_p}{\Lambda_k}O_1(1)\right\}, \tag{3.13}$$

Conducting the analogous considerations for the function $V_k(x)$ at $x \in G_l$ and $k \in \mathbf{N}$ we'll have

$$\overline{V_k(x)} = \left(2\operatorname{Im}\Lambda_k\right)^{1/q}\left\{G_{k,l}^{(1)}\exp\left[i\Lambda_k\left(x-\xi_{l-1}\right)\right] + \right.$$

$$\left. + G_{k,l}^{(2)}\exp\left[i\Lambda_k\left(\xi_l - x\right)\right] + \frac{\left\|U_k\right\|_q}{\Lambda_k}O_2(1)\right\}, \tag{3.14}$$

where $G_{k,l}^{(j)}$ $(j = 1, 2)$ are some complex numbers, moreover

$$\left|G_{k,l}^{(j)}\right| \le C\left\|V_k\right\|_q \quad (j = 1, 2;\ k \in N), \tag{3.15}$$

and $O_2(1)$ is a bounded function from k, x and l.

It is easy to note that if N bounded sequences $\left\{a_n^{(r)}\right\}_{n=1}^{\infty}$ $\left(r=\overline{1,\mathbf{N}}\right)$ is given, then there exists a strongly increasing sequence of natural numbers $\{n_\nu\}_{\nu=1}^{\infty}$ such that each of the sequences $\left\{a_{n_\nu}^{(r)}\right\}_{\nu=1}^{\infty}$ $\left(r=\overline{1,\mathbf{N}}\right)$ converges.

Consequently, according to (3.11) and (3.15) we can assume (if it is necessary passing to the subsequence) that at $j=1,2$ and $l=\overline{1,m}$

$$\lim_{k\to\infty} H_{k,l}^{(j)} \left\|U_k\right\|_p^{-1} = h_l^{(j)}, \tag{3.16}$$

$$\lim_{k\to\infty} G_{k,l}^{(j)} \left\|V_k\right\|_q^{-1} = g_l^{(j)}, \tag{3.17}$$

are true, where $h_l^{(j)},\ g_l^{(j)}$ are some numbers.

Assume that

$$\beta_{k,l}^{(j)} = \left[H_{k,l}^{(1)} G_{k,l}^{(j)} + H_{k,l}^{(2)} G_{k,l}^{(3-j)}\right] \left\|U_k\right\|_p^{-1} \left\|V_k\right\|_q^{-1}, \tag{3.18}$$

$$\beta_{k,l}^{(2+j)} = \left[H_{k,l}^{(1)} G_{k+1,l}^{(j)} + H_{k,l}^{(2)} G_{k+1,l}^{(3-j)}\right] \left\|U_k\right\|_p^{-1} \left\|V_{k+1}\right\|_q^{-1}, \tag{3.19}$$

$$\beta_{k,l} = \left[\left|H_{k,l}^{(1)}\right| + \left|H_{k,l}^{(2)}\right|\right] \left\|U_k\right\|_p^{-1}, \tag{3.20}$$

$$\beta_{k,l}^{*} = \left[\left|G_{k,l}^{(1)}\right| + \left|G_{k,l}^{(2)}\right|\right] \left\|V_k\right\|_q^{-1}, \tag{3.21}$$

where $j=1,2;\ k\in\mathbf{N},\ l\in\mathbf{N}$. According to (3.11) and (3.15)-(3.21) we have

$$\lim_{k\to\infty} \beta_{k,l}^{(1)} = \lim_{k\to\infty} \beta_{k,l}^{(3)} = h_l^{(1)} g_l^{(1)} + h_l^{(2)} g_l^{(2)}, \tag{3.22}$$

$$\sum_{j=1}^{4}\sum_{l=1}^{m+1} \left|\beta_{k,l}^{(j)}\right| + \sum_{l=1}^{m+1} \beta_{k,l} + \sum_{l=1}^{m+1} \beta_{k,l}^{*} \le C, k\in N. \tag{3.23}$$

Using the introduced above notation by the immediate calculations we can see that at $k\in\mathbf{N}$ and $l=\overline{1,m+1}$ the

$$\frac{(U_k,V_k)}{\left\|U_k\right\|_p \left\|V_k\right\|_q} = \frac{\sum\limits_{l=1}^{m+1} (U_k,V_k)_{G_l}}{\left\|U_k\right\|_p \left\|V_k\right\|_q} = \frac{\operatorname{Im}\Lambda_k}{i\Lambda_k} \sum_{l=1}^{m+1} \beta_{k,l}^{(1)} \left\{\exp\left[2i\Lambda_k\left(\xi_l-\xi_{l-1}\right)\right]-1\right\} +$$

$$+2\operatorname{Im}\Lambda_k \sum_{l=1}^{m+1} \beta_{k,l}^{(2)} \left(\xi_l-\xi_{l-1}\right) \exp\left[i\Lambda_k\left(\xi_l-\xi_{l-1}\right)\right] + \frac{\sum\limits_{l=1}^{m+1} \beta_{k,l}}{\Lambda_k} O(1) +$$

$$+\frac{\sum\limits_{l=1}^{m+1} \beta_{k,l}^{*}}{\Lambda_k} O(1) + \frac{\operatorname{Im}\Lambda_k}{\Lambda_k^2} O(1), \tag{3.24}$$

$$\frac{(U_k,V_{k+1})}{\left\|U_k\right\|_p \left\|V_{k+1}\right\|_q} = \frac{\sum\limits_{l=1}^{m+1} (U_k,V_{k+1})_{G_l}}{\left\|U_k\right\|_p \left\|V_{k+1}\right\|_q} = 2\left(\operatorname{Im}\Lambda_k\right)^{1/p} \left(\operatorname{Im}\Lambda_{k+1}\right)^{1/q} \times$$

$$\times\left\{\frac{1}{i\left(\Lambda_{k+1}+\Lambda_k\right)}\sum_{l=1}^{m+1}\beta_{k,l}^{(3)}\left(\exp\left[i\left(\xi_i-\xi_{l-1}\right)\left(\Lambda_{k+1}-\Lambda_k\right)\right]-1\right)+\right.$$

$$+\frac{1}{i\left(\Lambda_{k+1}-\Lambda_k\right)}\sum_{l=1}^{m+1}\beta_{k,l}^{(4)}\left(\exp\left[i\Lambda_{k+1}\left(\xi_l-\xi_{l-1}\right)\right]-\exp\left[i\Lambda_k\left(\xi_l-\xi_{l-1}\right)\right]\right)+$$

$$\left.+\frac{\sum\limits_{l=1}^{m+1}\beta_{k,l}}{\Lambda_{k+1}\operatorname{Im}\Lambda_k}O(1)+\frac{\sum\limits_{l=1}^{m+1}\beta_{k,l}^{*}}{\Lambda_k\operatorname{Im}\Lambda_{k+1}}O(1)+\frac{O(1)}{\Lambda_k\Lambda_{k+1}}\right\} \tag{3.25}$$

are true, where $O(1)$ means the bounded function from k not necessarily the same.

Since $\{U_k(x)\}_1^\infty$ is a basis of the space $L_p(G)$ and $\{V_k(x)\}_1^\infty$ is a system biorthogonally conjugated to $\{U_k(x)\}_1^\infty$, then it is well-known (see for example [4, p.370]) that

$$1\le\|U_k\|_p\|V_k\|_q\le C,\quad k\in N.$$

Consequently, not loosing generality, we can assume (if it is necessary passing to the subsequence) that it holds

$$\lim_{k\to\infty}\|U_k\|_p\|V_k\|_q=\alpha, \tag{3.26}$$

where $\alpha\ge1$.

Note, that $(U_k,V_k)=1\quad(k\in\mathbf{N})$. From here and from (3.23)-(3.24) we conclude that at $k\in N$ the relation

$$\frac{1}{\|U_k\|_p\|V_k\|_q}=-\frac{\operatorname{Im}\Lambda_k}{i\Lambda_k}\sum_{l=1}^{m+1}\beta_{k,l}^{(1)}+\frac{O(1)}{\operatorname{Im}\Lambda_k}.$$

is true.

Consequently, by virtue of (3.26) we have

$$\lim_{k\to\infty}\frac{\operatorname{Im}\Lambda_k}{i\Lambda_k}\sum_{l=1}^{m+1}\beta_{k,l}^{(1)}=-\alpha^{-1}\ne0.$$

Allowing for the last relation and (3.22) we obtain

$$\lim_{k\to\infty}\sum_{l=1}^{m+1}\beta_{k,l}^{(1)}=\sum_{l=1}^{m+1}\left(h_l^{(1)}g_l^{(1)}+h_l^{(2)}g_l^{(2)}\right)\equiv\beta\ne0, \tag{3.27}$$

$$\lim_{k\to\infty}\frac{\operatorname{Im}\Lambda_k}{i\Lambda_k}=-(\alpha\beta)^{-1}\ne0. \tag{3.28}$$

Since $(U_k,V_{k+1})=0\quad(k\in\mathbf{N})$, then from (3.24) we have

$$0=\sum_{l=1}^{m+1}\beta_{k,l}^{(3)}\left(\exp\left[i\left(\xi_l-\xi_{l-1}\right)\left(\Lambda_{k+1}-\Lambda_k\right)\right]-1\right)+$$

$$+\frac{\Lambda_{k+1}+\Lambda_k}{\Lambda_{k+1}-\Lambda_k}\sum_{l=1}^{m+1}\beta_{k,l}^{(4)}\left(\exp\left[i\Lambda_{k+1}\left(\xi_l-\xi_{l-1}\right)\right]-\exp\left[i\Lambda_k\left(\xi_l-\xi_{l-1}\right)\right]\right)+$$

$$+\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)}{\Lambda_{k+1}\operatorname{Im}\Lambda_k}\left(\sum_{l=1}^{m+1}\beta_{k,l}\right)O(1)+\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)\left(\sum\limits_{l=1}^{m+1}\beta_{k,l}^{*}\right)O(1)}{\Lambda_k\operatorname{Im}\Lambda_{k+1}}+$$

$$+\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)}{\Lambda_k\Lambda_{k+1}}O(1). \tag{3.29}$$

Further, according to (3.4), (3.5), (3.6*) and (3.28) at $k\in\mathbf{N}$ we have

$$\exp\left[i\left(\xi_l-\xi_{l-1}\right)\left(\Lambda_{k+1}-\Lambda_k\right)\right]=\frac{O(1)}{\Lambda_k},$$

$$\frac{\Lambda_{k+1}+\Lambda_k}{\Lambda_{k+1}-\Lambda_k}\exp\left[i\Lambda_{k+1}\left(\xi_l-\xi_{l-1}\right)\right]-\exp\left[i\Lambda_k\left(\xi_l-\xi_{l-1}\right)\right]=\frac{O(1)}{\Lambda_k}$$

$$\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)}{\Lambda_{k+1}\operatorname{Im}\Lambda_k}=\frac{O(1)}{\Lambda_k},\quad\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)}{\Lambda_k\operatorname{Im}\Lambda_{k+1}}=\frac{O(1)}{\Lambda_k},\quad\frac{i\left(\Lambda_{k+1}+\Lambda_k\right)}{\Lambda_k\Lambda_{k+1}}=\frac{O(1)}{\Lambda_k}.$$

By virtue of (3.29) and the last three relations we have

$$0=-\sum_{l=1}^{m+1}\beta_{k,l}^{(3)}+O(1)/\Lambda_k.$$

Hence, it follows that $\lim\limits_{k\to\infty}\sum\limits_{l=1}^{m+1}\beta_{k,l}^{(3)}=0$. Then by virtue of (3.22) we have $\lim\limits_{k\to\infty}\sum\limits_{l=1}^{m+1}\beta_{k,l}^{(1)}=0$, that contradicts to (3.27). Theorem 2 is proved.

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Nazim B. Kerimov, Elchin J. Ibadov
Baku State University
23, Z.Khalilov str., AZ1149, Baku, Azerbaijan
Tel.: (99412) 438 21 54 (off.),
(99412) 430 01 17 (apt.)

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