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MATHEMATICS

The Basis Properties of Eigenfunctions in the Eigenvalue Problem with a Spectral Parameter in the Boundary Condition

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Consider the eigenvalue problem

$$y^{IV} - (q(x)y')' = \lambda y, \quad 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{2b}$$

$$y(l)\cos\gamma + y''(l)\sin\gamma = 0, \tag{2c}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{2d}$$

where λ is the spectral parameter; $Ty \equiv y''' - qy'$; $q(x)$ is a absolutely continuous positive function on the interval $[0, l]$; and α, β, γ, a, b, c, and d are real constants such that $0 \le \alpha, \beta, \gamma \le \frac{\pi}{2}$.

In what follows, we assume that

$$\sigma = bc - ad > 0.$$

Boundary value problems for second- and fourth-order ordinary differential operators with a spectral parameter in the boundary conditions have been extensively studied (see, e.g., [1–9]). In [3–5], such problems were associated with particular physical processes.

The basis properties of the system of eigenfunctions in the Sturm–Liouville problem with a spectral parameter in the boundary conditions were studied in various function spaces in [7–9]. The existence of eigenvalues, estimate for eigenvalues and eigenfunctions, and expansion theorems for fourth-order operators with a spectral parameter in the boundary condition were considered in [1, 6].

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This paper deals with the basis properties in $L_p(0, l)$ $(1 < p < \infty)$ of the system of eigenfunctions of boundary value problem (1), (2).

1. ASYMPTOTIC FORMULAS FOR EIGENVALUES AND EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEM (1), (2)

We introduce the boundary condition

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \quad \delta \in [0, \pi/2]. \tag{2d'}$$

and, along with problem (1), (2), consider boundary value problem (1), (2a)–(2c), (2d').

Theorem 1 [10]. *The eigenvalues of boundary value problem* (1), (2a)–(2c), (2d') *are simple and form an infinitely increasing sequence* $0 \le \mu_1(\delta) < \mu_2(\delta) < \ldots < \mu_n(\delta) < \ldots$. *Moreover, the eigenfunction* $v_n^{(\delta)}(x)$ *corresponding to the eigenvalue* $\mu_n(\delta)$ *has exactly* $n - 1$ *simple zeros in the interval* $(0, l)$.

The numbers τ, ρ, τ_n, ρ_n $(n \in \mathbb{N})$ and the function $z(x, t)$, $x \in [0, l]$, $t \in \mathbb{R}$ are defined as

$$\tau = \begin{cases} \dfrac{3(1 + \operatorname{sgn}\beta + \operatorname{sgn}\delta)}{4} - 1, \text{ if } \operatorname{sgn}\alpha + \operatorname{sgn}\gamma = 1, \\ \dfrac{5 + 2\operatorname{sgn}\alpha}{4} + ((-1)^{\operatorname{sgn}\alpha + \operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}\alpha + \operatorname{sgn}\delta}) \\ \times \dfrac{6\operatorname{sgn}\alpha - 3}{8} - 1, \quad \text{if } \operatorname{sgn}\alpha + \operatorname{sgn}\gamma \ne 1, \end{cases}$$

$$\rho = \begin{cases} \dfrac{3(1 + \operatorname{sgn}\beta + \operatorname{sgn}|c|)}{4}, \text{ if } \operatorname{sgn}\alpha + \operatorname{sgn}\gamma = 1, \\ \dfrac{5 + 2\operatorname{sgn}\alpha}{4} + ((-1)^{\operatorname{sgn}\alpha + \operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}\alpha + \operatorname{sgn}|c|}) \\ \times \dfrac{6\operatorname{sgn}\alpha - 3}{8}, \quad \text{if } \operatorname{sgn}\alpha + \operatorname{sgn}\gamma \ne 1, \end{cases}$$

$$\tau_n = (n-\tau)\frac{\pi}{l}, \quad \rho_n = (n-\rho)\frac{\pi}{l},$$

$$z(x,t) = \begin{cases} \sin\left(tx + \frac{\pi}{2}\operatorname{sgn}\beta\right) \\ -\cos\left(tl + \frac{\pi}{2}(\operatorname{sgn}\beta + \operatorname{sgn}\gamma)\right)\exp(-t(l-x)), \\ \text{if } \operatorname{sgn}\alpha + \operatorname{sgn}\beta = 1, \\ \sin tx - \cos tx + (-1)^{\operatorname{sgn}\alpha}\exp(-tx) \\ + \sqrt{2}\times(-1)^{1-\operatorname{sgn}\gamma}\sin\left(tl + \frac{\pi}{4}\times(-1)^{\operatorname{sgn}\gamma}\right) \\ \times\exp(-t(l-x)), \text{ if } \operatorname{sgn}\alpha + \operatorname{sgn}\beta \neq 1. \end{cases}$$

Theorem 2. *It holds that*

$$\sqrt[4]{\mu_n(\delta)} = \tau_n + O\left(\frac{1}{n}\right), \tag{3}$$

$$v_n^{(\delta)}(x) = z(x, \tau_n) + O\left(\frac{1}{n}\right). \tag{4}$$

The proof of Theorem 2 is based on Theorem 1 and formulas (45.a) and (45.b) in [11].

Theorem 3. *The spectrum of boundary value problem* (1), (2) *consists of an infinite sequence of simple eigenvalues* $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ *such that* $\lambda_n > 0$ *for* $n \geq 3$. *For the eigenvalues* λ_n *and the corresponding eigenfunctions* $y_n(x)$, *we have the asymptotic formulas*

$$\sqrt[4]{\lambda_n} = \rho_n + O\left(\frac{1}{n}\right), \tag{5}$$

$$y_n(x) = z(x, \rho_n) + O\left(\frac{1}{n}\right). \tag{6}$$

The existence of eigenvalues of boundary value problem (1), (2) follows from Theorem 1 and the lemma below.

Lemma 1. *Let* $y(x, \lambda)$ *be a nontrivial solution to problem* (1), (2a)–(2c).

Then, in each interval $(\mu_{n-1}(0), \mu_n(0))$, *where* $n \in \mathbb{N}$ *and* $\mu_0(0) = -\infty$, *the function* $\frac{Ty(l,\lambda)}{y(l,\lambda)}$ *is continuous and strictly increasing. Moreover*,

$$\lim_{\lambda \to -\infty} \frac{Ty(l,\lambda)}{y(l,\lambda)} = -\infty.$$

The proofs of asymptotic formulas (5) and (6) are based on Theorem 1 in [11], Theorem 2, and the following lemma.

Lemma 2. *For sufficiently large* $n \in \mathbb{N}$,

$$\mu_{n-2}(0) < \mu_{n-1}\left(\frac{\pi}{2}\right) < \lambda_n < \mu_{n-1}(0),$$

$$\textit{if} \quad c \neq 0 \ \ u \ \ \frac{a}{c} \geq 0,$$

$$\mu_{n-2}(0) < \lambda_n < \mu_{n-1}\left(\frac{\pi}{2}\right) < \mu_{n-1}(0),$$

$$\textit{if} \quad c \neq 0 \ \ u \ \ \frac{a}{c} < 0,$$

$$\mu_{n-1}(0) < \lambda_n < \mu_n\left(\frac{\pi}{2}\right) < \mu_n(0), \quad \textit{if} \quad c = 0.$$

2. BASIS PROPERTY IN $L_p(0, l)$ $(1 < p < \infty)$ OF THE SYSTEM OF EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEM (1), (2)

Theorem 4. *Let r be an arbitrary fixed positive integer.*

Then the system $\{y_n(x)\}$ $(n = 1, 2, \ldots; n \neq r)$ *is minimal in* $L_p(0, l)$ $(1 < p < \infty)$.

Proof sketch of Theorem 4. It is sufficient to prove the existence of a system $\{u_n(x)\}$ $(n = 1, 2, \ldots; n \neq r)$ that is biorthogonal adjoint to $\{y_n(x)\}$ $(n = 1, 2, \ldots; n \neq r)$.

Let $c = 0$. The biorthogonal adjoint system is given by the relation

$$u_n(x) = \left(y_n(x) - \frac{y_n(l)}{y_r(l)}y_r(x)\right)\Big/\left(\|y_n\|_2^2 - \frac{a}{d}y_n^2(l)\right), \tag{7}$$

where $\|\cdot\|_p$ denotes the norm in $L_p(0, l)$.

Let $c \neq 0$. The number N is determined by the inequality $\mu_{N-1}(0) < -\frac{d}{c} \leq \mu_N(0)$. When $\lambda_{N+1} \neq -\frac{d}{c}$, the biorthogonal adjoint system is

$$u_n(x) = \left(y_n(x) - \frac{(c\lambda_r + d)y_n(l)}{(c\lambda_n + d)y_r(l)}y_r(x)\right)\Big/\left(\|y_n\|_2^2 + \frac{\sigma y_n^2(l)}{(c\lambda_n + d)^2}\right). \tag{8}$$

When $\lambda_{N+1} = -\frac{d}{c}$, the biorthogonal adjoint system is given by

$$u_n(x) = \left(y_n(x) - \frac{\sigma y_n(l)}{c(c\lambda_n + d)Ty_{N+1}(l)}y_{N+1}(x)\right)\Big/\left(\|y_n\|_2^2 + \frac{\sigma y_n^2(l)}{(c\lambda_n + d)^2}\right) \tag{9}$$

for $r = N+1$, by (8) for $r \neq N+1$ and $n \neq N+1$, and by

$$u_n(x) = \frac{y_{N+1}(x) - \dfrac{c(c\lambda_r + d)Ty_{N+1}(l)}{\sigma y_r(l)} y_r(x)}{\|y_{N+1}\|^2 + \dfrac{c^2(Ty_{N+1}(l))^2}{\sigma}}$$

for $r \neq N+1$ and $n = N+1$.

Theorem 4 is proved.

Taking into account (5) and (6), we find from (7)–(9) the asymptotic formula

$$u_n(x) = l^{-1} y_n(x) + O\left(\frac{1}{n}\right). \tag{10}$$

Below is the main result of this paper.

Theorem 5. *Let r be an arbitrary fixed positive integer.*

Then the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) *forms a basis in* $L_p(0, l)$ ($1 < p < \infty$), *and this basis is unconditional for* $p = 2$.

Proof sketch of Theorem 5. The boundary conditions (2a)–(2c), (2d') are strongly regular (see [11]). Then, by Theorem 5.1 in [12], the system of eigenfunctions $\{v_n^{(\delta)}(x)\}_{n=1}^{\infty}$ of problem (1), (2a)–(2c), (2d') forms a basis in $L_p(0, l)$ ($1 < p < \infty$) and this basis is unconditional for $p = 2$.

Let $c = 0$. We compare the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) with $\{v_n^{(0)}(x)\}_{n=1}^{\infty}$. By virtue of (4) and (6), for sufficiently large n, it holds that

$$\|y_{n+1}(x) - v_n^{(0)}(x)\|_2 \le \text{const} \cdot n^{-1},$$

which implies the convergence of the series

$$\sum_{n=1}^{r-1} \|y_n(x) - v_n^{(0)}(x)\|_2^2 + \sum_{n=r}^{\infty} \|y_{n+1}(x) - v_n^{(0)}(x)\|_2^2$$

(for $r = 1$, the first sum is absent). Consequently, $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) is quadratically close to the system $\{v_n^{(0)}(x)\}_{n=1}^{\infty}$. By Theorem 4, the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) is minimal in $L_p(0, l)$ ($1 < p < \infty$). Then, by Theorem 9.9.8 in [13], the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) is an unconditional basis in $L_2(0, l)$.

The case $c \neq 0$ is considered in a similar manner, with the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) compared with $\{v_n^{(\pi/2)}(x)\}_{n=1}^{\infty}$.

Let $\tilde{v}_n^{(\delta)}(x) = v_n^{(\delta)}(x)\|v_n^{(\delta)}(x)\|_2^{-1}$ ($n = 1, 2, \ldots$). Since boundary value problem (1), (2a)–(2c), (2d") is self-adjoint, it follows from (4) that the systems $\{\tilde{v}_n^{(0)}(x)\}_{n=1}^{\infty}$ = 1 and $\{v_n^{(\pi/2)}(x)\}_{n=1}^{\infty}$ are uniformly bounded orthonormal bases in $L_2(0, l)$.

By using (4), (6), and (10), it is easy to see that

$$\begin{aligned} y_n(x) &= l^{1/2}\tilde{v}_n(x) + O\left(\frac{1}{n}\right), \\ u_n(x) &= l^{-1/2}\tilde{v}_n(x) + O\left(\frac{1}{n}\right) \end{aligned} \tag{11}$$

for $n \in \mathbb{N}$ and $n \neq r$. Here,

$$\tilde{v}_n(x) = \begin{cases} \tilde{v}_n^{(0)}(x) \text{ for } c = 0 \text{ and } 1 \le n \le r-1; \\ \tilde{v}_{n-1}^{(0)}(x) \text{ for } c = 0 \text{ and } n \ge r-1; \\ \tilde{v}_n^{(\pi/2)}(x) \text{ for } c \neq 0 \text{ and } 1 \le n \le r-1; \\ \tilde{v}_{n-1}^{(\pi/2)}(x) \text{ for } c \neq 0 \text{ and } n \ge r+1. \end{cases}$$

We now fix $p \in (1, 2)$. Since $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) is a basis in $L_2(0, l)$, this system is complete in $L_p(0, l)$. Using (11) and Theorem 2.3 in [14], we can prove the estimate

$$\left\| \sum_{n=1, n \neq r}^{k} (f, u_n) y_n \right\|_p \le M_p \|f\|_p, \quad k = 1, 2, \ldots$$

for an arbitrary function $f(x)$ in $L_p(0, l)$, where M_p is a positive constant. By Theorem 6 in [15], the system $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) is a basis in $L_p(0, l)$.

Let $2 < p < +\infty$. Following the above line of reasoning, we prove the basis property of $\{u_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) in $L_q(0, l)$ $\left(q = \dfrac{p}{p-1}\right)$, which is equivalent to the basis property of $\{y_n(x)\}$ ($n = 1, 2, \ldots; n \neq r$) in $L_p(0, l)$ (see [15]). Theorem 5 is proved.

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