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ON THE UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS FOR A SPECTRAL PROBLEM WITH A BOUNDARY CONDITION RATIONALLY DEPENDING ON THE EIGENPARAMETER

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Abstract. The spectral problem

$$-y'' + q(x)y = \lambda y, 0 < x < 1,$$
$$y(0)\cos\beta = y'(0)\sin\beta, 0 \le \beta < \pi; \frac{y'(1)}{y(1)} = h(\lambda),$$

is considered, where λ is a spectral parameter, $q(x)$ is real-valued continuous function on $[0,1]$ and

$$h(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k},$$

with the real coefficients and $a \ge 0, b_k > 0, c_1 < c_2 < \cdots < c_N, N \ge 0$.

The sharpened asymptotic formulae for eigenvalues and eigenfunctions of above-mentioned spectral problem are obtained and the uniform convergence of the spectral expansions of the continuous functions in terms of eigenfunctions are presented.

1. Introduction

Consider the spectral problem

$$-y'' + q(x)y = \lambda y, \ 0 < x < 1, \tag{1.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta, \ 0 \le \beta < \pi, \tag{1.2}$$

$$y'(1)/y(1) = h(\lambda), \tag{1.3}$$

where λ is a spectral parameter, $q(x)$ is real-valued continuous function on $[0,1]$ and

$$h(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k},$$

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with the real coefficients and $a \geq 0,\ b_k > 0,\ c_1 < c_2 < \cdots < c_N,\ N \geq 0$. When $h(\lambda) = \infty$, then the boundary condition (1.3) is interpreted as a Dirichlet boundary condition $y(1) = 0$. The case $N = 0$ makes $h(\lambda)$ as affine by λ.

As is known, solutions that obtained by using the Fourier method of partial differential equations are represented by a series. Therefore, the investigation of the convergence of these series is of great importance. In this study, we investigate the uniform convergence properties of the Fourier series expansions in terms of eigenfunctions of the boundary value problem (1.1)-(1.3).

It was proven in [2] that the eigenvalues of the problem (1.1)-(1.3) are real, simple and form a sequence $\lambda_0 < \lambda_1 < \cdots$ accumulating only at $+\infty$ and with $\lambda_0 < c_1$. Moreover, if ω_n is the number of zeros in the interval (0,1) of the eigenfunction y_n, corresponding to eigenvalue λ_n, then $\omega_n = n - m_n$, where m_n is the number of points $c_i \leq \lambda_n$. In particular, $\omega_0 = 0$ and $\omega_n = n - N$ when $\lambda_n > c_N$. Further, following asymptotic formulae will be valid for sufficiently large n:

$$\lambda_n = \left((n+\nu)\,\pi\right)^2 + O(1), \tag{1.4}$$

$$\nu = \begin{cases} -\frac{1}{2} - N & \text{if } a \neq 0, \beta \neq 0, \\ -N & \text{if } a \neq 0, \beta = 0, \\ -N & \text{if } a = 0, \beta \neq 0, \\ \frac{1}{2} - N & \text{if } a = 0, \beta = 0. \end{cases} \tag{1.5}$$

It was proven in [11] that if $a \neq 0$ and $k_0, k_1, \ldots, k_N$ are pairwise different nonnegative integers, then the system

$$y_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N) \tag{1.6}$$

is a basis in $L_p(0,1)\,(1 < p < \infty)$; moreover if $p = 2$, then this basis is unconditional. If $a = 0$ and $k_1, k_2, \ldots, k_N$ are pairwise different nonnegative integers, then the system

$$y_n(x)\,(n = 0, 1, \ldots; n \neq k_1, k_2, \ldots, k_N) \tag{1.7}$$

is a basis in $L_p(0,1)\,(1 < p < \infty)$; moreover if $p = 2$, then this basis is unconditional. Further, if $a \neq 0$ and $\lambda_n \neq c_j$; for all $n = 0, 1, \ldots$ and $j = 1, \ldots, N$, then the system $u_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ which is biorthogonally conjugate to the system (1.6) is defined by

$$u_n(x) = \frac{A_{n,k_0,\ldots,k_N}(x)}{B_n \Delta}, \tag{1.8}$$

where

$$A_{n,k_0,\ldots,k_N}(x) = \begin{vmatrix} y_n(x) & y_n(1) & \frac{y_n(1)}{\lambda_n - c_1} & \cdots & \frac{y_n(1)}{\lambda_n - c_N} \\ y_{k_0}(x) & y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{k_N}(x) & y_{k_N}(1) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}, \tag{1.9}$$

$$B_n = \|y_n\|^2 + \left(a + \sum_{k=1}^{N} \frac{b_k}{(\lambda_n - c_k)^2}\right) y_n^2(1), \tag{1.10}$$

$$\Delta = \begin{vmatrix} y_{k_0}(1) & \frac{y_{k_0}(1)}{\lambda_{k_0}-c_1} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0}-c_N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k_N}(1) & \frac{y_{k_N}(1)}{\lambda_{k_N}-c_1} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N}-c_N} \end{vmatrix}. \tag{1.11}$$

Suppose that some of the numbers $c_j\,(j = 1, \ldots, N)$ are eigenvalues of the problem (1.1)-(1.3). For example, $\lambda_{k_t} = c_s$ for some t and s. Then, all the elements in $(t{+}2)$th row of the determinant (1.9) vanish, except the first element and $(s{+}2)$th element; the first element $y_{k_t}(x)$ does not change but $\frac{y_{k_t}(1)}{\lambda_{k_t}-c_s}$ is replaced by $-\frac{y'_{k_t}(1)}{b_s}$. Moreover, $B_{k_t} = \|y_{k_t}\|^2 + \frac{(y'_{k_t}(1))^2}{b_s}$.

If $a = 0$, then we construct the system $u_n(x)\,(n = 0, 1, \ldots; n \neq k_1, \ldots, k_N)$ which is biorthogonally conjugate the system (1.7) with an obvious modification. In particular, we obtain the corresponding determinant $A_{n,k_1,\ldots,k_N}(x)$ of degree $N + 1$ from the determinant (1.9) by deleting the second row and the second column.

Many authors investigated the spectral properties of the problem (1.1)-(1.3) in the special cases. For example, the basis properties in $L_p(0,1)\,(1 < p < \infty)$ of the boundary value problem

$$\begin{gathered} -y'' = \lambda y,\ 0 < x < 1, \\ y(0) = 0, (a - \lambda)y'(1) = b\lambda y(1), \end{gathered} \tag{1.12}$$

where a, b are positive constants, was given in [8]. The uniform convergence of the spectral expansions in the systems of eigenfunctions of the problem (1.12) and the problem

$$\begin{gathered} y'' + \lambda y = 0, \\ y(0) = 0, y'(1) = d\lambda y(1), d > 0 \end{gathered}$$

was obtained in [9]. The basis properties in $L_2(0,1)$ of the eigenfunctions of boundary value problem

$$\begin{gathered} -y'' + q(x)y = \lambda y,\ 0 < x < 1 \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)\, y(1) = (c_1\lambda + d_1)\, y'(1), \end{gathered} \tag{1.13}$$

where $q(x)$ is a real-valued continuous function on $[0,1]$ and $|b_0| + |d_0| \neq 0$, $a_1 d_1 - b_1 c_1 > 0$, was studied more detail in [15]. The uniform convergence of the Fourier series expansions in terms of eigenfunctions of the problem (1.13) was researched in [14].

One can find many articles, for example [3], [4], [5, 6, 7, 10], [12], [13], [17], where studied the uniform convergence of the spectral expansions.

Note that the case $N = 0$ is a special case of the problem (1.13), with $c_1 = 0$. Henceforth, we assume that $N \geq 1$.

2. Main results

In this section, we will give sharpened asymptotic formulae for eigenvalues and eigenfunctions and investigate the uniform convergence of the Fourier expansions for the continuous functions in the system of eigenfunctions of the problem (1.1)-(1.3).

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ denote the solutions of the equation (1.1) that satisfy the initial conditions

$$\varphi(0,\lambda)=1,\ \varphi'(0,\lambda)=\widetilde{h}, \tag{2.1}$$

$$\psi(0,\lambda)=0,\ \psi'(0,\lambda)=1, \tag{2.2}$$

where $\widetilde{h}$ is an arbitrary real number.

Theorem 2.1. *Let $\lambda_n=s_n^2$. The following asymptotic formulae are valid for sufficiently large n:*

$$s_n=(n+\nu)\pi+\frac{A_{a,\beta}}{n\pi}+O\left(\frac{\delta_{n,\nu}}{n}\right); \tag{2.3}$$

$$\begin{aligned} y_n(x) &= \psi(x,\lambda_n) \\ &= \frac{\sin(n+\nu)\pi x}{(n+\nu)\pi}+\frac{\alpha_a x-\frac{1}{2}\int_0^x q(\tau)\,d\tau}{(n\pi)^2}\cos(n+\nu)\pi x \\ &\quad +\frac{\cos(n+\nu)\pi x}{2(n\pi)^2}\int_0^x q(\tau)\cos 2(n+\nu)\pi\tau d\tau \\ &\quad +\frac{\sin(n+\nu)\pi x}{2(n\pi)^2}\int_0^x q(\tau)\sin 2(n+\nu)\pi\tau d\tau \\ &\quad +O\left(\frac{\delta_{n,\nu}}{n^2}\right),\ \text{if}\ \ \beta=0; \end{aligned} \tag{2.4}$$

$$\begin{aligned} y_n(x) &= \varphi(x,\lambda_n) \\ &= \cos(n+\nu)\pi x \\ &\quad +\frac{\widetilde{h}-(\widetilde{h}+\alpha_a)x+\frac{1}{2}\int_0^x q(\tau)\,d\tau}{n\pi}\sin(n+\nu)\pi x \\ &\quad +\frac{\sin(n+\nu)\pi x}{2n\pi}\int_0^x q(\tau)\cos 2(n+\nu)\pi\tau d\tau \\ &\quad -\frac{\cos(n+\nu)\pi x}{2n\pi}\int_0^x q(\tau)\sin 2(n+\nu)\pi\tau d\tau \\ &\quad +O\left(\frac{\delta_{n,\nu}}{n}\right),\ \text{if}\ \ 0<\beta<\pi; \end{aligned} \tag{2.5}$$

where ν *is defined by* (1.5), $\widetilde{h} = \cot\beta$, $A_{a,\beta} = \begin{cases} \alpha_a, & \text{if } \beta = 0, \\ \widetilde{h} + \alpha_a, & \text{if } 0 < \beta < \pi, \end{cases}$

$\alpha_a = \begin{cases} -b + \frac{1}{2}\int_0^1 q(\tau)\,d\tau, & \text{if } a = 0, \\ \frac{1}{a} + \frac{1}{2}\int_0^1 q(\tau)\,d\tau, & \text{if } a \neq 0 \end{cases}$ *and* $\delta_{n,\nu} = \left|\int_0^1 q(\tau)\cos 2(n+\nu)\pi\tau d\tau\right| + \frac{1}{n}$.

Proof. We will prove only the case of $a \neq 0$ and $0 < \beta < \pi$. The other cases are proven similarly. From (1.4) and (1.5),

$$s_n = \sqrt{\lambda_n} = \left(n - \frac{1}{2} - N\right)\pi + \varepsilon_n \tag{2.6}$$

is satisfied, where $\varepsilon_n = O\left(n^{-1}\right)$.

Let $\lambda = s^2$. From (2.1), the equality

$$\varphi(x,\lambda) = \cos sx + \frac{\widetilde{h}}{s}\sin sx + \frac{1}{s}\int_0^x q(\tau)\,\varphi(\tau,\lambda)\sin s(x-\tau)\,d\tau \tag{2.7}$$

is obtained [16, ChapterI, Section 1.2, Lemma 1.2.1].

Let $s = \sigma + it$. Then there exists $s_0 > 0$ such that for $|s| > s_0$, the estimate

$$\varphi(x,\lambda) = \cos sx + O\left(e^{|t|x}|s|^{-1}\right) \tag{2.8}$$

is valid [16, Chapter I, Section 1.2, Lemma 1.2.2], where the function

$$O\left(e^{|t|x}|s|^{-1}\right)$$

is the entire function of s for any fixed x in $[0,1]$. Moreover, (2.8) holds uniformly by x for $0 \le x \le 1$.

The formulae (2.6)-(2.8) yield the following:

$$\begin{aligned} y_n(x) &= \varphi(x,\lambda_n) \\ &= \cos s_n x + \frac{\widetilde{h}}{s_n}\sin s_n x + \frac{\sin s_n x}{2s_n}\int_0^x q(\tau)\,d\tau \\ &\quad + \frac{\sin s_n x}{2s_n}\int_0^x q(\tau)\cos 2s_n\tau d\tau - \frac{\cos s_n x}{2s_n}\int_0^x q(\tau)\sin 2s_n\tau d\tau \\ &\quad + O\left(n^{-2}\right). \end{aligned} \tag{2.9}$$

Because

$$\cos s_n x = \cos\left(n - \frac{1}{2} - N\right)\pi x + O\left(n^{-1}\right),$$
$$\sin s_n x = \sin\left(n - \frac{1}{2} - N\right)\pi x + O\left(n^{-1}\right),$$

the equality (2.9) can be written as

(2.10)
$$\begin{aligned}\varphi(x,\lambda_n) = {} & \cos s_n x + \frac{\widetilde{h}\sin\left(n-\frac{1}{2}-N\right)\pi x}{n\pi} + \frac{\sin\left(n-\frac{1}{2}-N\right)\pi x}{2n\pi}\int_0^x q(\tau)\,d\tau \\ & + \frac{\sin\left(n-\frac{1}{2}-N\right)\pi x}{2n\pi}\int_0^x q(\tau)\cos(2n-1-2N)\pi\tau d\tau \\ & - \frac{\cos\left(n-\frac{1}{2}-N\right)\pi x}{2n\pi}\int_0^x q(\tau)\sin(2n-1-2N)\pi\tau d\tau + O\left(n^{-2}\right).\end{aligned}$$

In addition, by differentiating equality (2.7) with respect to x and substituting equality (2.6), we obtain the estimate

$$\varphi'(x,\lambda_n) = -s_n \sin s_n x + O(1). \tag{2.11}$$

The function $\varphi(x,\lambda)$ satisfies the boundary condition (1.2). Therefore, the eigenvalues of the problem (1.1)-(1.3) satisfy the equation

$$\varphi'(1,\lambda_n) = \left(a\lambda_n + b - \sum_{k=1}^{N}\frac{b_k}{\lambda_n - c_k}\right)\varphi(1,\lambda_n). \tag{2.12}$$

By using (2.10) and (2.11), we obtain the estimates

$$\varphi(1,\lambda_n) = (-1)^{n-N}\varepsilon_n + \frac{(-1)^{n-N-1}\left(\widetilde{h}+\frac{1}{2}\int_0^1 q(\tau)\,d\tau\right)}{n\pi} + O\left(\frac{\delta_{n,\nu}}{n}\right),$$
$$\varphi'(1,\lambda_n) = (-1)^{n-N}\left(n-\frac{1}{2}-N\right)\pi + O(1).$$

Substituting last estimates in the equation (2.12), we obtain the equality

$$\begin{aligned} & (-1)^{n-N}\left(n-\frac{1}{2}-N\right)\pi + O(1) \\ = {} & \left(a\left(n-\frac{1}{2}-N\right)^2\pi^2 + O(1)\right) \\ & \times\left((-1)^{n-N}\varepsilon_n + \frac{(-1)^{n-N-1}\left(\widetilde{h}+\frac{1}{2}\int_0^1 q(\tau)\,d\tau\right)}{n\pi} + O\left(\frac{\delta_{n,\nu}}{n}\right)\right).\end{aligned}$$

The last equation shows that the estimate

$$\varepsilon_n = \frac{\widetilde{h}+\frac{1}{a}+\frac{1}{2}\int_0^1 q(\tau)\,d\tau}{n\pi} + O\left(\frac{\delta_{n,\nu}}{n}\right) \tag{2.13}$$

is valid. The equality (2.3) is proven in case of $a \neq 0$ and $0 < \beta < \pi$.

We obtain the estimate

$$\begin{aligned}\cos s_n x = {} & \cos\left(n-\tfrac{1}{2}-N\right)\pi x - \tfrac{\left(\widetilde{h}+\frac{1}{a}+\frac{1}{2}\int_0^1 q(\tau)d\tau\right)x}{n\pi} \\ & \times\sin\left(n-\tfrac{1}{2}-N\right)\pi x + O\left(\tfrac{\delta_{n,\nu}}{n}\right),\end{aligned} \tag{2.14}$$

by using (2.6) and (2.13). The equality (2.5) follows from (2.10) and (2.14).

The proof of Theorem 2.1 is completed. □

Let $a \neq 0$; $\lambda_n \neq c_j$ for all $n = 0, 1, \ldots$; $j = 1, \ldots, N$ and $f(x) \in C[0,1]$. We define the determinant

$$\Delta' = \begin{vmatrix} (f, y_{k_0}) & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_1} & \cdots & \frac{y_{k_0}(1)}{\lambda_{k_0} - c_N} \\ \vdots & \vdots & \ddots & \vdots \\ (f, y_{k_N}) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}, \tag{2.15}$$

where (f, g) denotes the inner product of $f(x)$ and $g(x)$ in space $L_2(0,1)$.

If $\lambda_{k_t} = c_s$ for some t and s, then all the elements $(t{+}1)$th row of the determinant (2.15) vanish, except the first element and $(s{+}1)$th element; the first element (f, y_{k_t}) does not change, but the $(s{+}1)$th element $\frac{y_{k_t}(1)}{\lambda_{k_t} - c_s}$ is replaced by $-\frac{y'_{k_t}(1)}{b_s}$.

Theorem 2.2. *Suppose that $k_0, k_1, \ldots, k_N$ are pairwise different nonnegative integers and $f(x) \in C[0,1]$.*

I. *Let $a = 0$ and $\beta = 0$. If the function $f(x)$ has a uniformly convergent Fourier series expansion in the system $\left\{\sqrt{2}\sin\left(n - \frac{1}{2}\right)\pi x\right\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_1, \ldots, k_N)$ and this expansion is uniformly convergent on $[0,1]$.*

II. *Let $a \neq 0$ and $\beta = 0$. If the function $f(x)$ has a uniformly convergent Fourier series expansion in the system $\left\{\sqrt{2}\sin n\pi x\right\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ and this expansion is uniformly convergent on every interval $[0,b]$, $0 < b < 1$. The Fourier series of $f(x)$ in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ is uniformly convergent on $[0,1]$ if and only if the determinant Δ' vanishes.*

III. *Let $a = 0$ and $0 < \beta < \pi$. If the function $f(x)$ has a uniformly convergent Fourier series expansion in the system $\left\{\sqrt{2}\cos n\pi x\right\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_1, \ldots, k_N)$ and this expansion is uniformly convergent on $[0,1]$.*

IV. *Let $a \neq 0$ and $0 < \beta < \pi$. If the function $f(x)$ has a uniformly convergent Fourier series expansion in the system $\left\{\sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x\right\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ and this expansion is uniformly convergent on every interval $[0,b]$, $0 < b < 1$. The Fourier series of $f(x)$ in the system $y_n(x)\,(n = 0, 1, \ldots; n \neq k_0, k_1, \ldots, k_N)$ is uniformly convergent on $[0,1]$ if and only if the determinant Δ' vanishes.*

Proof. We will only prove the first and the second case. Other cases are proven similarly.

The first case: Let $\lambda_n \neq c_j, j = 1, \ldots, N, n = 0, 1, \ldots$. Consider the Fourier series $f(x)$ on the interval $[0,1]$ in the system $y_n(x)$, $(n = 0, 1, \ldots;\ n \neq k_1, \ldots, k_N)$:

$$F(x) = \sum_{n=0, n\neq k_1,\ldots,k_N}^{\infty} (f, u_n)\, y_n(x), \tag{2.16}$$

where

$$u_n(x) = \frac{A_{n,k_1,\ldots,k_N}(x)}{B_n \Delta}, \tag{2.17}$$

$$A_{n,k_1,\ldots,k_N}(x) = \begin{vmatrix} y_n(x) & \frac{y_n(1)}{\lambda_n - c_1} & \cdots & \frac{y_n(1)}{\lambda_n - c_N} \\ y_{k_1}(x) & \frac{y_{k_1}(1)}{\lambda_{k1} - c_1} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k_N}(x) & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \cdots & \frac{y_{k_N}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}, \tag{2.18}$$

$$B_n = \|y_n\|^2 + y_n^2(1) \sum_{k=1}^{N} \frac{b_k}{(\lambda_n - c_k)^2}, \tag{2.19}$$

$$\Delta = \begin{vmatrix} \frac{y_{k_1}(1)}{\lambda_{k_1} - c_1} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_1} - c_N} \\ \vdots & \ddots & \vdots \\ \frac{y_{k_N}(1)}{\lambda_{k_N} - c_1} & \cdots & \frac{y_{k_1}(1)}{\lambda_{k_N} - c_N} \end{vmatrix}. \tag{2.20}$$

Note that the series (2.16) is uniformly convergent on $[0,1]$ if and only if the series

$$F_1(x) = \sum_{n=r+1}^{\infty} (f, u_n)\, y_n(x) \tag{2.21}$$

is uniformly convergent on $[0,1]$, where $r = \max\{k_1, k_2, \ldots, k_N\}$.

By virtue of (2.3), (2.4), (2.17)-(2.20), the equality

$$(f, u_n) = \frac{(f, A_{n,k_1,\ldots,k_N})}{B_n \Delta} = \frac{(f, y_n)}{B_n} + O\left(n^{-3}\right) \tag{2.22}$$

holds. From (2.3), (2.4) and (2.19), the estimate

$$B_n = \frac{1}{2(n\pi)^2} + O\left(n^{-3}\right)$$

is valid. Therefore, the equality (2.22) can be written as

$$(f, u_n) = 2(n\pi)^2 (f, y_n) + (f, y_n)\, O(n). \tag{2.23}$$

By using (2.4) and (2.23), we have

$$(f,u_n)\, y_n(x) = \left(f, \sqrt{2}\sin\left(n+\frac{1}{2}-N\right)\pi x\right)\sqrt{2}\sin\left(n+\frac{1}{2}-N\right)\pi x + R_{n,1}(x),$$

where

$$\begin{aligned} R_{n,1}(x) = &\left(f, \sin\left(n+\frac{1}{2}-N\right)\pi x\right) O\left(n^{-1}\right) \\ &+ \left(f(x)\,\alpha_1(x), \cos\left(n+\frac{1}{2}-N\right)\pi x\right) O\left(n^{-1}\right) \\ &+ \left(f, \alpha_{n,1}(x)\cos\left(n+\frac{1}{2}-N\right)\pi x\right) O\left(n^{-1}\right) \\ &+ \left(f, \beta_{n,1}(x)\sin\left(n+\frac{1}{2}-N\right)\pi x\right) O\left(n^{-1}\right) \\ &+ O\left(\frac{\delta_{n,\nu}}{n}\right), \end{aligned} \tag{2.24}$$

$$\alpha_1(x) = \alpha_a x - \frac{1}{2}\int_0^x q(\tau)\, d\tau,$$

$$\alpha_{n,1}(x) = \int_0^x q(\tau)\cos(2n+1-2N)\,\pi\tau d\tau,$$

$$\beta_{n,1}(x) = \int_0^x q(\tau)\sin(2n+1-2N)\,\pi\tau d\tau.$$

By virtue of (2.24), we obtain

$$\begin{aligned} |R_{n,1}(x)| \le &\frac{\text{const}}{n}\left\{\left|\left(f, \sin\left(n+\frac{1}{2}-N\right)\pi x\right)\right| \right. \\ &+ \left|\left(f(x)\,\alpha_1(x), \cos\left(n+\frac{1}{2}-N\right)\pi x\right)\right| \\ &+ \left|\left(f, \alpha_{n,1}(x)\cos\left(n+\frac{1}{2}-N\right)\pi x\right)\right| \\ &\left. + \left|\left(f, \beta_{n,1}(x)\sin\left(n+\frac{1}{2}-N\right)\pi x\right)\right| + \delta_{n,\nu}\right\} \\ \le &\text{ const}\left\{\left|\left(f, \sin\left(n+\frac{1}{2}-N\right)\pi x\right)\right|^2 \right. \\ &+ \left|\left(f(x)\,\alpha_1(x), \cos\left(n+\frac{1}{2}-N\right)\pi x\right)\right|^2 \\ &\left. + \left(\int_0^1 |f(x)\,\alpha_{n,1}(x)|\, dx\right)^2 + \left(\int_0^1 |f(x)\,\beta_{n,1}(x)| dx\right)^2 + \frac{\delta_{n,\nu}}{n}\right\} \end{aligned}$$

for sufficiently large n. The numerical series

$$\sum_{n=r+1}^{\infty}\left|\left(f,\sin\left(n+\frac{1}{2}-N\right)\pi x\right)\right|^2,$$

$$\sum_{n=r+1}^{\infty}\left|\left(f\left(x\right)\alpha_1\left(x\right),\cos\left(n+\frac{1}{2}-N\right)\pi x\right)\right|^2,$$

$$\sum_{n=r+1}^{\infty}\frac{\delta_{n,\nu}}{n}$$

are convergent. On the other hand, by virtue of Bessel inequality, we obtain

$$\begin{aligned}
&\sum_{n=r+1}^{\infty}\left(\int_0^1\left|f\left(x\right)\alpha_{n,1}\left(x\right)\right|dx\right)^2\\
&\leq\|f\|^2\sum_{n=r+1}^{\infty}\int_0^1\left|\alpha_{n,1}\left(x\right)\right|^2dx\\
&=\|f\|^2\int_0^1\sum_{n=r+1}^{\infty}\left|\int_0^x q\left(\tau\right)\cos\left(2n+1-2N\right)\pi\tau d\tau\right|^2dx\\
&\leq\text{const}\|f\|^2\int_0^1\int_0^x\left|q\left(\tau\right)\right|^2d\tau dx\leq\text{const}\|f\|^2\|q\|^2.
\end{aligned}$$

Similarly, we obtain the estimate

$$\sum_{n=r+1}^{\infty}\left(\int_0^1\left|f\left(x\right)\beta_{n,1}\left(x\right)\right|dx\right)^2\leq\text{const}\|f\|^2\|q\|^2.$$

Consequently, the series

$$\sum_{n=r+1}^{\infty}R_{n,1}\left(x\right)$$

is absolutely and uniformly convergent on $[0,1]$.

If some of numbers $c_j\ (j=1,\ldots,N)$ are eigenvalues of the problem (1.1)-(1.3), then the proof is completely similar.

The first case is proven.

The second case: Let $\lambda_n\neq c_j, j=1,\ldots,N, n=0,1,\ldots$. Consider the Fourier series $f(x)$ on the interval $[0,1]$ in the system $y_n\left(x\right)$ ($n=0,1,\ldots$; $n\neq k_0,k_1,\ldots,k_N$):

$$G\left(x\right)=\sum_{n=0,n\neq k_0,k_1,\ldots,k_N}^{\infty}\left(f,u_n\right)y_n\left(x\right), \tag{2.25}$$

where $u_n\left(x\right)$ is defined by (1.8).

Note that the series (2.25) is uniformly convergent on $[0,1]$ if and only if the series

$$G_1(x) = \sum_{n=r}^{\infty} (f, u_n)\, y_n(x) \tag{2.26}$$

is uniformly convergent on $[0,1]$, where $r = \max\{k_0, k_1, k_2, \ldots, k_N\}$.

By virtue of (1.8)-(1.11), (2.3), (2.4), the equality

$$(f, u_n) = \frac{(f, y_n)}{B_n} - \frac{\Delta^{'}}{\Delta}\frac{y_n(1)}{B_n} + O\left(n^{-4}\right) \tag{2.27}$$

holds, where Δ' is defined by (2.15).

The series $\sum_{n=r}^{\infty} \frac{(f,y_n)}{B_n} y_n(x)$ is uniformly convergent on $[0,1]$. This can be seen by the method of the first case. By virtue of (2.27), if $\Delta^{'} = 0$, then the second part of this case is truth.

Let $\Delta^{'} \neq 0$. From (1.10), (2.3) and (2.4), the estimates

$$y_n(1) = \frac{(-1)^{n-N}}{2a(n\pi)^2} + O\left(\frac{\delta_{n,\nu}}{n^2}\right),$$

$$B_n = \frac{1}{2(n\pi)^2} + O\left(n^{-3}\right)$$

holds. Therefore, the equality

$$\sum_{n=r}^{\infty} \frac{y_n(1)}{B_n} y_n(x) = -\sum_{n=r-N}^{\infty} \frac{\sin n\pi(1+x)}{n} + \sum_{n=r}^{\infty} O\left(\frac{\delta_{n,\nu}}{n}\right)$$

is valid. The series $\sum_{n=r}^{\infty} O\left(\frac{\delta_{n,\nu}}{n}\right)$ is absolutely and uniformly convergent on $[0,1]$. On the other hand, the series

$$\sum_{n=r}^{\infty} \frac{\sin n\pi t}{n}$$

is uniformly convergent on every closed interval $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$ [1, Chapter I, Section 30, Theorem1]. So, the series

$$\sum_{n=r-N}^{\infty} \frac{\sin n\pi(1+x)}{n}$$

is uniformly convergent on every closed interval [0,b] $0 < b < 1$.

The second case is proven.

The proof of theorem 2.2 is completed. $\square$

3. Example

Consider the problem

$$-y'' = \lambda y, 0 < x < 1, \tag{3.1}$$

$$y\,(0) = 0, \frac{y'\,(1)}{y\,(1)} = a\lambda - \frac{\pi^2}{\lambda - \pi^2}, \tag{3.2}$$

where a is a positive number.

$\lambda_r = 0$ and $\lambda_s = \pi^2$ are eigenvalues of the problems (3.1) and (3.2), where r and s are certain non-negative integers and $y_r(x) = x$ and $y_s(x) = \sin \pi x$ are corresponding eigenfunctions, respectively.

Let $f(x) = (x^2 - x)(x^2 + mx + n)$ and $g(x) = (x^2 - x)(5x - 3)$, where $m = 11 - \frac{120}{\pi^2}$ and $n = \frac{72}{\pi^2} - 7$. Since $f(0) = f(1) = g(0) = g(1) = 0$, then $(f, \sin n\pi x) = O(n^{-3})$ and $(g, \sin n\pi x) = O(n^{-3})$. On the other hand, $(f, y_r) = (f, y_s) = 0$ and $(g, y_r) = 0$, $(g, y_s) = \frac{2}{\pi^3}$.
From here and (2.15), we obtain

$$\Delta^{'} = \begin{vmatrix} (g, y_r) & \frac{y_r(1)}{\lambda_r - c_1} \\ (g, y_s) & -\frac{y'_s(1)}{b_1} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{\pi^2} \\ \frac{2}{\pi^3} & \frac{1}{\pi} \end{vmatrix} = \frac{2}{\pi^5} \neq 0.$$

Consequently, by Theorem 2.2 the Fourier series of $f\,(x)$ and $g\,(x)$ in the system $y\,(x)\,(n = 0, 1, \ldots; n \neq r, s)$ are uniformly convergent on $[0, 1]$ and on $[0, b]$ $(0 < b < 1)$, respectively.

References

[1] N. K. Bary,*A Treatise on Trigonometric Series*, Vol. I, Pergemon Press, Oxford, 1964.

[2] P. A. Binding, P. J. Browne, and B. A. Watson, *Sturm-Liouville problems with boundary conditions rationally dependent on the parameter. I*, Proc. Edinb. Math. Soc. **45** (2002), 631–645.

[3] D. A. Gulyaev, *On the uniform convergence of spectral expansions for a spectral problem with boundary conditions of the third kind one of which contains the spectral parameter*, Differ. Equ. **47** (2011), no. 10, 1503–1507.

[4] ———, *On the uniform convergence in W_2^m of spectral expansions for a spectral problem with boundary conditions of the third kind one of which contains the spectral parameter*, Differ. Equ. **48** (2012), no. 10, 1450–1453.

[5] N. Yu. Kapustin, *On the uniform convergence of the Fourier Series for a spectral problem with squared spectral parameter in a boundary condition*, Differ. Equ. **46** (2010), no. 10, 1504–1507.

[6] ———, *On the uniform convergence in C^1 of Fourier Series for a spectral problem with squared spectral parameter in a boundary condition*, Differ. Equ. **47** (2011), no. 10, 1394–1399.

[7] ———, *On the spectral problem arising in the solution of a mixed problem for the heat equation with a mixed derivative in the boundary condition*, Differ. Equ. **48** (2012), no. 5, 694–699.

[8] N. Yu. Kapustin and E. I. Moiseev, *The basis property in L_p of the systems of eigenfunctions corresponding two problems with a spectral parameter in the boundary conditions*, Differ. Equ. **36** (2000), no. 10, 1498–1501.

[9] ———, *Convergence of spectral expansions for functions of the Hölder class for two problems with spectral parameter in the boundary condition*, Differ. Equ. **36** (2000), 1182–1188.

[10] ———, *A remark on the convergence problem for spectral expansions corresponding to a classical problem with spectral parameter in the boundary condition*, Differ. Equ. **37** 2001, no. 12, 1677–1683.

[11] N. B. Kerimov and Y. N. Aliyev, *The basis property in L_p of the boundary value problem rationally dependent on the eigenparameter*, Studia Math. **174** (2006), no. 2, 201–212.

[12] N. B. Kerimov, S. Goktas, and E. A. Maris, *Uniform convergence of the spectral expansions in terms of root functions for a spectral problem*, Electron. J. Differential Equations **2016** (2016), no. 80, 14 pp.

[13] N. B. Kerimov and E. A. Maris, *On the basis properties and convergence of expansions in terms of eigenfunctions for a spectral problem with a spectral parameter in the boundary condition*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2014), 245–258.

[14] ———, *On the uniform convergence of the Fourier series for one spectral problem with a spectral parameter in a boundary condition*, Math. Meth. Appl. Sci. (2015), DOI: 10.1002/mma.3640.

[15] N. B. Kerimov and V. S. Mirzoev, *On the basis properties of one spectral problem with a spectral parameter in a boundary condition*, Siberian Math. J. **44** (2003), no. 1, 813–816.

[16] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac operators*, Kluwer Academic Publishers, Netherlands, 1991.

[17] D. B. Marchenkov, *On the convergence of spectral expansions of functions for problems with a spectral parameter in a boundary condition*, Differ. Equ. **41** (2005), no. 10, 1419–1422.

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