DIRECT AND INVERSE PROBLEMS FOR THE HEAT EQUATION WITH A DYNAMIC TYPE BOUNDARY CONDITION

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Abstract. This paper considers the initial-boundary value problem for the heat equation with a dynamic type boundary condition. Under some regularity, consistency and orthogonality conditions, the existence, uniqueness and continuous dependence upon the data of the classical solution are shown by using the generalized Fourier method. This paper also investigates the inverse problem of finding a time-dependent coefficient of the heat equation from the data of integral overdetermination condition.

1. Introduction

Let $T>0$ be a fixed number and $D_T=\{(x,t):0<x<1,\,0<t\le T\}$.

Consider the following initial-boundary value problem for the heat equation in $\bar{D}_T$:

$$u_t = u_{xx} - p(t)u + f(x,t), \tag{1.1}$$

with the initial condition

$$u(x,0)=\varphi(x), \tag{1.2}$$

and the boundary conditions

$$u(0,t)=0, \quad au_{xx}(1,t)+du_x(1,t)-bu(1,t)=0, \tag{1.3}$$

where f, φ are given functions when $0\le t\le T$, $0\le x\le 1$ and a,b,d are given numbers.

When the coefficient $p(t)$, $0\le t\le T$ is also given, the problem of finding $u(x,t)$ from using equation (1.1), initial condition (1.2) and boundary conditions (1.3) is termed as the direct (or forward) problem.

This problem can be used in a heat transfer and diffusion processes where a source parameter is present. Taking into account the equation at $x=1$, in the case $a\ne 0$, the second boundary condition becomes to the form of dynamically boundary condition as

$$au_t(1,t)+du_x(1,t)+(ap(t)-b)u(1,t)=af(1,t).$$

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This boundary condition is observed in the process of cooling of a thin solid bar one end of which is placed in contact in the case of perfect thermal contact [1, p. 262]. Another possible application of such boundary condition is announced in [2, p. 79] which represents a boundary reaction in diffusion of chemical, where the term $du_x(1,t)$ represents the diffusive transport of materials to the boundary.

When the function $p(t)$, $0 \le t \le T$ is unknown, the inverse problem formulates a problem of finding a pair of functions $\{p(t),\ u(x,t)\}$ such that satisfy the equation (1.1), initial condition (1.2), boundary conditions (1.3) and overdetermination condition

$$\int_0^1 u(x,t)dx = E(t), 0 \le t \le T. \tag{1.4}$$

If we let $u(x,t)$ present the temperature distribution, then above-mentioned inverse problem can be regarded as a control problem with source control. The source control parameter $p(t)$ needs to be determined from thermal energy $E(t)$.

Because the function p is space independent, a, b and d are constants and the boundary conditions are linear and homogeneous, the method of separation of variables is suitable for studying the problems under consideration. It is well known that, the main difficulty for applying Fourier method is its basisness, i.e. expansion in terms of eigenfunctions of auxiliary spectral problem

$$\begin{cases} y''(x) + \lambda y(x) = 0,\ 0 \le x \le 1, \\ y(0) = 0, \\ (a\lambda + b)\, y(1) = dy^{'}(1). \end{cases} \tag{1.5}$$

In contrast to classical Sturm-Liouville problem, this problem has spectral parameter also in boundary condition. It makes impossible to apply the classical results to the expansion in terms of eigenfunctions [3,4]. The spectral analysis of such type of problems started by Walter [5]. The important developments are made by Fulton [6], Kerimov, Allakhverdiev [7,8], Binding, Browne, Seddighi [9], Kapustin, Moiseev [10], Kerimov, Poladov [11]. It is useful to note the paper [12] for which the results on expansion in terms of eigenfunctions are firmly used in present paper.

The inverse problem of finding the coefficient $p(t)$ in the equation (1.1) with the nonlocal boundary conditions are considered in the papers [13-16]. In contrast to these papers, in present paper the boundary conditions are localized to the points $x = 0$ and $x = 1$. The literature devoted to inverse problems of a finding time-dependent coefficient for the equation (1.1) with the localized boundary conditions are so vast, see [17-19], to name only a few references. The principal difference the boundary conditions in present paper from the others localized boundary conditions is that the existence the term $u_{xx}(1,t)$. As is noted, this boundary condition is reduced to a dinamical type boundary condition by using the expression of the equation (1.1).

The paper is organized as follows. In Chapter 2, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced. In Chapter 3, the existence, uniqueness and the continuous dependence upon the data of the solution of direct problem (1.1)-(1.3) is proved. Finally in Chapter 4, the existence, uniqueness and continuous dependence upon the data of the solution of the inverse problem (1.1)-(1.4) is shown.

2. Some properties of the auxiliary spectral problem

Consider the spectral problem (1.4) with $ad > 0$. It is known in [9] that, eigenvalues λ_n, $n = 0, 1, 2, ...$ are real and simple. They are unbounded increasing sequence

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots ,$$

and the eigenfunction $y_n(x)$ corresponding to λ_n has exactly n simple zeros in the interval $(0, 1)$.

For the positive eigenvalues $\lambda_n = \mu_n^2$, μ_n are the simple positive roots of characteristic equation $(\frac{a}{d}\mu^2 + \frac{b}{d})\sin\mu = \mu\cos\mu$. The placement of all eigenvalues with respect to $\lambda = 0$ are as follows:

$$\begin{aligned} \lambda_0 &< 0 < \lambda_1 < \lambda_2 < \cdots , \text{ for } \frac{b}{d} > 1, \\ \lambda_0 &= 0 < \lambda_1 < \lambda_2 < \cdots , \text{ for } \frac{b}{d} = 1, \\ 0 &< \lambda_0 < \lambda_1 < \lambda_2 < \cdots , \text{ for } \frac{b}{d} < 1 \end{aligned}$$

In the case $\frac{b}{d} > 1$, the eigenfunctions are $y_0(x) = e^{\mu_0 x} - e^{-\mu_0 x}$, $y_n(x) = \sin(\mu_n x)$, $n = 1, 2, ...$, which correspond to eigenvalues $\lambda_0 = -\mu_0^2$, $\lambda_n = \mu_n^2, n = 1, 2, ...$ Besides, $\pi n < \mu_n < \frac{\pi}{2} + \pi n$.

In the case $\frac{b}{d} = 1$, the eigenfunctions are $y_0(x) = x$, $y_n(x) = \sin(\mu_n x)$, $n = 1, 2, ...$, which correspond to eigenvalues $\lambda_0 = 0$, $\lambda_n = \mu_n^2, n = 1, 2, ...$ Besides, $\pi n < \mu_n < \frac{\pi}{2} + \pi n$.

In the case $\frac{b}{d} < 1$, the eigenfunctions are $y_n(x) = \sin(\mu_n x)$, $n = 0, 1, 2, ...$, which correspond to eigenvalues $\lambda_n = \mu_n^2$, $n = 0, 1, 2, ...$ In addition, $\pi n < \mu_n < \frac{\pi}{2} + \pi n$ for $0 \le \frac{b}{d} < 1$ and $\pi n < \mu_n < \pi + \pi n$ for $\frac{b}{d} < 0$.

It is easy to see that, in all of these cases

$$\int_0^1 y_n(x)dx > 0, \ n = 0, 1, 2, ... \tag{2.1}$$

The detailed investigation of the characteristic equation allows us to determine the following asymptotic of eigenfunctions and eigenvalues:

$$\begin{aligned} \mu_n &= \pi n + \frac{d}{a\pi n} + O(\frac{1}{n^3}), \\ y_n(x) &= \sin(\pi n x) + \left[\frac{d}{a\pi n}\cos(\pi n x) + O(\frac{1}{n^3})\right] x, \end{aligned} \tag{2.2}$$

for a sufficiently large n.

Let n_0 be arbitrary fixed nonnegative integer. It is shown in [12] that the system of eigenfunctions $\{y_n(x)\}$ $(n = 0, 1, 2, ...; n \ne n_0)$ is a Riesz basis for $\mathbf{L}_2[0, 1]$. The system $\{u_n(x)\}$ $(n = 0, 1, 2, ...; n \ne n_0)$, biortogonal to the system $\{y_n(x)\}$ $(n = 0, 1, 2, ...; n \ne n_0)$ has the form

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_{n_0}(1)} y_{n_0}(x)}{\|y_n\|^2_{\mathbf{L}_2[0,1]} + \frac{a}{d} y_n^2(1)},$$

that is $(y_n, u_m) = \delta_{n,m}$, $n, m = 0, 1, 2, ...; n, m \ne n_0$ where $\delta_{n,m}$ is the Kronecker symbol.

The following lemma is true for the systems $\{y_n(x)\}$, $\{u_n(x)\}$ $(n = 0, 1, 2, ...; n \neq n_0)$.

Lemma 1. *Let $\varphi(x) \in \mathbf{C}^3\,[0,1]$ be arbitrary function satisfying the conditions*

$$\varphi(0) = \varphi''(0) = 0,\ \ \varphi(1) = \varphi'(1) = \varphi''(1) = 0 \tag{2.3}$$

and

$$\int_0^1 \varphi(x) y_{n_0}(x) dx = 0. \tag{2.4}$$

Then the inequalities

$$\sum_{\substack{n=0\\(n\neq n_0)}}^{\infty} |\lambda_n(\varphi, y_n)| \ \leq \ C_0 \left\|\varphi\right\|_{\mathbf{C}^3[0,1]}, \ \sum_{\substack{n=0\\(n\neq n_0)}}^{\infty} |\lambda_n(\varphi, u_n)| \leq C \left\|\varphi\right\|_{\mathbf{C}^3[0,1]} \tag{2.5}$$

(C_0 and C are constants)

hold.

Proof. The asymptotic representation (2.2) implies

$$\|y_n\|^2_{\mathbf{L}_2[0,1]} = \frac{1}{2} + O(\frac{1}{n^2}),\ \ y_n^2(1) = O(\frac{1}{n^2}).$$

Then there exist constants α and β, such that

$$\beta > \|y_n\|^2_{\mathbf{L}_2[0,1]} + \frac{a}{d} y_n^2(1) > \alpha > 0.$$

According to last inequality, under condition (2.4), the convergence of series $\sum\limits_{\substack{n=0\\(n\neq n_0)}}^{\infty} |\lambda_n(\varphi, y_n)|$ is equivalent to convergence of $\sum\limits_{\substack{n=0\\(n\neq n_0)}}^{\infty} |\lambda_n(\varphi, u_n)| = \sum\limits_{\substack{n=0\\(n\neq n_0)}}^{\infty} \frac{|\lambda_n(\varphi, y_n)|}{\|y_n\|^2_{\mathbf{L}_2[0,1]} + \frac{a}{d} y_n^2(1)}$. In addition, this inequality allows us to obtain second estimate in (2.5) if we know first one.

Since $\lambda_n y_n = -y_n''$ and $y_n(0) = 0$, the equality $\lambda_n(\varphi, y_n) = -(\varphi'', y_n)$ is obtained with helping two time integration by parts and using (2.3). Taking into account the representation (2.2) we obtain

$$\lambda_n(\varphi, y_n) = -(\varphi'', \sin(\pi n x)) - \frac{d}{a\pi n}(x\varphi'', \cos(\pi n x)) + \left\|\varphi''\right\|_{\mathbf{C}[0,1]} O(\frac{1}{n^2}). \tag{2.6}$$

By using Schwarz and Bessel inequalities, it is easy to show that

$$\begin{aligned} \sum_{\substack{n=0\\(n\neq n_0)}}^{\infty} \frac{d}{a\pi n} \left|(x\varphi'', \cos(\pi n x))\right| \ &\leq \ const \left\|x\varphi''\right\|_{\mathbf{L}_2[0,1]} \leq const \left\|\varphi''\right\|_{\mathbf{C}[0,1]}, \\ \sum_{\substack{n=1\\(n\neq n_0)}}^{\infty} \left\|\varphi''\right\|_{\mathbf{C}[0,1]} O(\frac{1}{n^2}) \ &\leq \ const \left\|\varphi''\right\|_{\mathbf{C}[0,1]}. \end{aligned} \tag{2.7}$$

In addition, by using integration by parts and Schwarz and Bessel inequalities we obtain that

$$\sum_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} |(\varphi'', \sin(\pi n x))| \leq const \, \|\varphi'''\|_{\mathbf{C}[0,1]} \,. \tag{2.8}$$

Putting (2.6) and (2.7) in (2.8) yields the inequality (2.5). □

Let us introduce the following notation for the simplicity.

Notation 1. *The class of functions which satisfy the conditions of the Lemma 1 we will denote as $\mathbf{\Phi}_{n_0}$, that is,*

$$\mathbf{\Phi}_{n_0} \equiv \left\{ \begin{array}{c} \varphi(x) \in \mathbf{C}^3\,[0,1] : \varphi(0) = \varphi''(0) = 0,\ \varphi(1) = \varphi'(1) = \varphi''(1) = 0, \\ \int_0^1 \varphi(x) y_{n_0}(x) dx = 0 \end{array} \right\}$$

Because the series $\sum\limits_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} |\lambda_n(\varphi, y_n)|$ is majorant for the series $\sum\limits_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} |(\varphi, y_n)|$, the following corollary of Lemma 1 hold.

Corollary 1. *For arbitrary $\varphi(x) \in \Phi_{n_0}$ the estimates*

$$\sum_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} |(\varphi, y_n)| \leq \bar{C}_0 \, \|\varphi\|_{\mathbf{C}^3[0,1]} \,, \quad \sum_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} |(\varphi, u_n)| \leq \bar{C} \, \|\varphi\|_{\mathbf{C}^3[0,1]}$$

hold, where $\bar{C}_0$ and $\bar{C}$ are constants.

3. Classical solution of the direct problem

Let $p(t) \in \mathbf{C}[0,T]$ be a known continuous function. The function $u(x,t)$ from the class $\mathbf{C}^{2,0}\left(\bar{D}_T\right) \cap \mathbf{C}^{2,1}\left(D_T\right)$ that satisfy (1.1) in D_T, the initial condition (1.2) and the boundary condition (1.3) is said to be classical solution of the mixed problem (1.1)-(1.3).

The smoothness conditions $f \in \mathbf{C}\left(D_T\right),\ \varphi \in \mathbf{C}^2[0,1]$ and the consistency conditions

$$\varphi(0) = 0,\ a\varphi''(1) + d\varphi'(1) - b\varphi(1) = 0 \tag{3.1}$$

are the necessary conditions for the existence of a classical solution of the problem (1.1)-(1.3).

To construct the formal solution of the problem (1.1)-(1.3) we will use generalized Fourier method. In accordance with this method, the solution $u(x,t)$ is sought in a Fourier series in term of the eigenfunctions $\{y_n(x)\}$ $(n = 0,1,2,...; n \neq n_0)$ of auxiliary spectral problem (1.5):

$$u(x,t) = \sum_{\substack{n=0 \\ (n\neq n_0)}}^{\infty} v_n(t) y_n(x),\ v_n(t) = (u, u_n).$$

For the functions $v_n(t), n = 0,1,2,...; n \neq n_0$ we obtain the Cauchy problem

$$\begin{aligned} v_n'(t) + (p(t) + \lambda_n) v_n(t) &= f_n(t), \\ v_n(0) &= \varphi_n, \end{aligned}$$

where $f_n(t) = (f, y_n), \varphi_n = (\varphi, u_n)$.

Solving these Cauchy problems, we obtain

$$v_n(t) = \varphi_n e^{-\int_0^t [p(s)+\lambda_n]ds} + \int_0^t f_n(s) e^{-\int_s^t [p(\tau)+\lambda_n]d\tau} ds$$

and the formal solution of the mixed problem (1.1)-(1.3) is expressed via the series

$$u(x,t) = \sum_{\substack{n=1 \\ (n \neq n_0)}}^{\infty} \left[\varphi_n e^{-\lambda_n t - \int_0^t p(s)ds} + \int_0^t f_n(s) e^{-\lambda_n (t-s) - \int_s^t p(\tau)d\tau} ds \right] y_n(x). \tag{3.2}$$

Now we prove a theorem on the existence of the classical solution of the problem (1.1)-(1.3).

Theorem 1. *(Existence) If $p(t) \in \mathbf{C}[0,T]$, $f(x,t) \in \mathbf{C}\left(\bar{D}_T\right), \varphi(x) \in \mathbf{\Phi}_{n_0}$ and $f(x,t) \in \mathbf{\Phi}_{n_0}$ for every $t \in [0,T]$. Then the series (3.2) gives a classical solution of the problem (1.1)-(1.3) and $u(x,t)$ belongs to $\mathbf{C}^{2,1}\left(\bar{D}_T\right)$.*

Proof. By the hypotesis, we obtain from Lemma 1 that the series

$$M_1 \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} |\lambda_n \varphi_n| \text{ and } M_2 \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} |\lambda_n| \int_0^T |f_n(s)|\, ds \tag{3.3}$$

(M_1 and M_2 are some constants) are convergent. These series are the majorizing series of (3.2) and its $x-$partial, $xx-$partial derivatives. The majorizing series for the $x-$partial derivative of (3.2) follows from the fact that the functions $y_n'(x), n > 2$ have at least one zero in $(0,1)$, since the eigenfunction $y_n(x)$ corresponding to λ_n has exactly n simple zeros in the interval $(0,1)$. In this case, the equality $\lambda_n y_n = -y_n''$ implies $y_n'(x) = \lambda_n \int_x^\omega y_n(s)ds$, where $\omega \in (0,1)$ are a zero of $y_n'(x)$. Because the sequence $y_n(x), n = 1, 2, \ldots$ is uniformly bounded, the series $\sum\limits_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} |\varphi_n| \, |y_n'(x)|$ can be majorizing by $M_3 \sum\limits_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} |\varphi_n|$, where M_3 is some constant. Thus, the series (3.2) and its $x-$partial, $xx-$partial derivatives are uniformly convergent in $\bar{D}_T$. Since these series uniformly convergent, their sums $u(x,t)$, $u_x(x,t)$ and $u_{xx}(x,t)$ are continuos in $\bar{D}_T$. The series (3.3) are also majorizing series for the $t-$partial derivative of the (3.2). Therefore $u_t(x,t)$ also continuous in $\bar{D}_T$. Thus, $u(x,t) \in \mathbf{C}^{2,1}\left(\bar{D}_T\right)$ and satisfies the condition (1.1)-(1.3) by the superposition principle.

The proof of the theorem is complete. □

Remark 1. *The existence of the classical solution of the problem (1.1)-(1.3) can be obtained for each of the classes of functions $\mathbf{\Phi}_{n_0}$, $n_0 = 0, 1, 2, \ldots$, therefore, the classical solution exists for the union $\mathbf{\Phi} \equiv \bigcup\limits_{n_0=0}^{\infty} \mathbf{\Phi}_{n_0}$. It can be expected that the integral condition (2.4) is not essential for the existence of the classical solution.*

Remark 2. *The uniqueness of the solution of the problem (1.1)-(1.3), under the conditions of the Theorem 1 is obtained from the uniqueness of the representation*

(3.2). The uniqueness of the solution under smoothness and consistency conditions (3.1) is obtained by using maximum-minimum principle. It will be done in next theorem.

Lemma 2. *(maximum-minimum principle) Let* $p(t) \geq 0, t \in [0,T]$*, and* $u(x,t) \in \mathbf{C}^{2,0}\left(\bar{D}_T\right) \cap \mathbf{C}^{2,1}\left(D_T\right)$ *satisfies the equation (1.1) in* D_T*. If* $f(x,t) \leq 0$ *in* D_T *then*

$$u(x,t) \leq \max\left\{0, \max_{0\leq x\leq 1} u(x,0), \max_{0\leq t\leq T} u(0,t), \max_{0\leq t\leq T} u(1,t)\right\}.$$

If $f(x,t) \geq 0$ *in* D_T *then*

$$u(x,t) \geq \min\left\{0, \min_{0\leq x\leq 1} u(x,0), \min_{0\leq t\leq T} u(0,t), \min_{0\leq t\leq T} u(1,t)\right\}.$$

The proof of this lemma is omitted because it can be found in [20, p. 390].

Theorem 2. *(Uniqueness and continuous dependence upon the data)The classical solution of the problem (1.1)-(1.3), with* $p(t) \geq 0$ *is unique and depends continuously on* $f(x,t) \in \mathbf{C}\left(\bar{D}_T\right)$ *and* $\varphi(x) \in \mathbf{C}^2[0,1]$ *in the sense that*

$$\|u - \tilde{u}\|_{\mathbf{C}\left(\bar{D}_T\right)} \leq \|\varphi - \tilde{\varphi}\|_{\mathbf{C}[0,1]} + (1+|b|)T\left\|f - \tilde{f}\right\|_{\mathbf{C}\left(\bar{D}_T\right)}, \tag{3.4}$$

where $u(x,t)$ *and* $\tilde{u}(x,t)$ *are the classical solutions of (1.1)-(1.3) with the data* f, φ *and* $\tilde{f}, \tilde{\varphi}$*, respectively.*

Proof. Let $u(x,t)$ be the classical solution of the problem (1.1)-(1.3). Introduce the notations

$$R = \|f\|_{\mathbf{C}\left(\bar{D}_T\right)},\ K = \|\varphi\|_{\mathbf{C}[0,1]}.$$

Construct the function

$$g(x,t) = u(x,t) - Rt.$$

This function is a classical solution of the mixed problem

$$u_t = u_{xx} - p(t)u + f(x,t) - R - p(t)Rt,$$

$$u(x,0) = \varphi(x),$$

$$u(0,t) = -Rt,\ au_{xx}(1,t) + du_x(1,t) - bu(1,t) = bRt.$$

Allowing for $f(x,t) - R - p(t)Rt \leq 0$, $bRt \leq |b|\,RT$ and using maximum principle, we obtain the estimate $g(x,t) \leq \max\{K, |b|\,RT\}$, i.e.

$$u(x,t) \leq \max\{K, |b|\,RT\} + RT \leq K + (1+|b|)RT.$$

Similarly, if we introduce the function

$$h(x,t) = u(x,t) + Rt$$

and use the minimum principle, we arrive at a opposite estimate:

$$u(x,t) \geq -\max\{K, |b|\,RT\} - RT \geq -K - (1+|b|)RT.$$

Thus, if $u(x,t)$ is the classical solution of the problem (1.1)-(1.3) we have the estimate

$$\|u\|_{\mathbf{C}\left(\bar{D}_T\right)} \leq \|\varphi\|_{\mathbf{C}[0,1]} + (1+|b|)T\,\|f\|_{\mathbf{C}\left(\bar{D}_T\right)}. \tag{3.5}$$

The uniqueness follows from the fact that by virtue of the estimate (3.5), the homogeneous problem (1.1)-(1.3) (i.e. with $f = 0$, and $\varphi = 0$) has only a zero classical solution.

To prove continuous dependence on data we study the difference $v(x,t) = u(x,t) - \tilde{u}(x,t)$. This function is a classical solution of (1.1)-(1.3) with $f - \tilde{f}$, and $\varphi - \tilde{\varphi}$ substituted for f and φ, respectively. Applying the inequality (3.5) to $v(x,t)$ we arrive at the estimate (3.4). $\square$

4. Classical solution of the inverse problem

Let $p(t), t \in [0,T]$ be a unknown function. The pair $\{p(t), u(x,t)\}$ from the class $\mathbf{C}[0,T] \times (\mathbf{C}^{2,1}(\bar{D}_T) \cap \mathbf{C}^{2,0}(D_T))$ for which the conditions (1.1)–(1.4) are satisfied, is called a classical solution of the inverse problem (1.1)–(1.4).

We have the following assumptions on φ, E and f.

(A$_1$) (A$_1$)$_1$ $\varphi(x) \in \mathbf{\Phi}_{n_0}$; (A$_1$)$_2$ $\varphi_0 > 0, \varphi_n \geq 0, n = 0,1,2,...(n \neq n_0)$,

(A$_2$) (A$_2$)$_1$ $E(t) \in \mathbf{C}^1[0,T]$; $E(0) = \int\limits_0^1 \varphi(x)dx$;
(A$_2$)$_2$ $E(t) > 0, \forall t \in [0,T]$;

(A$_3$) (A$_3$)$_1$ $f(x,t) \in \mathbf{C}\left(\overline{D}_T\right)$; $f(x,t) \in \mathbf{\Phi}_{n_0}, \ \forall t \in [0,T]$;
(A$_3$)$_2$ $f_n(\tau) \geq 0, \ n = 0,1,2,...; n \neq n_0$;

where $\varphi_n = \int\limits_0^1 \varphi(x)u_n(x)dx, \ f_n(t) = \int\limits_0^1 f(x,t)u_n(x)dx, \ n = 0,1,2,....$

The main result is presented as follows.

Theorem 3. *(Existence and uniqueness) Let $(A_1) - (A_3)$ be satisfied. Then the inverse problem (1.1)-(1.4) has a unique classical solution.*

Proof. We already know that the solution of the mixed problem (1.1)-(1.3) is expressed via the series

(4.1)
$$u(x,t) = \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\varphi_n e^{-\lambda_n t - \int_0^t p(s)ds}\right] y_n(x) + \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\int_0^t f_n(s) e^{-\lambda_n (t-s) - \int_s^t p(\tau)d\tau} ds\right] y_n(x).$$

for arbitrary $p(t) \in \mathbf{C}[0,T]$. In addition $u(x,t) \in \mathbf{C}^{2,1}\left(\bar{D}_T\right)$.

Applying the overdetermination condition (1.4), we obtain the following Volterra integral equation of the second kind with respect to $q(t) = e^{\int\limits_0^t p(s)ds}$:

$$q(t) = F(t) + \int\limits_0^t K(t,\tau) q(\tau) d\tau, \tag{4.2}$$

where

$$
\begin{aligned}
F(t) &= \frac{1}{E(t)} \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\varphi_n e^{-\lambda_n t} \int_0^1 y_n(x) dx \right], \\
K(t,\tau) &= \frac{1}{E(t)} \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[f_{2n-1}(\tau) e^{-\lambda_n (t-\tau)} \int_0^1 y_n(x) dx \right].
\end{aligned} \tag{4.3}
$$

In the case of existence of the positive solution of (4.2) in class $\mathbf{C}^1[0,T]$, the function $p(t)$ can be determined from $q(t) = e^{\int_0^t p(s)ds}$ as

$$p(t) = \frac{q'(t)}{q(t)}. \tag{4.4}$$

By using the boundness of the sequence $\int_0^1 y_n(x)dx, n = 0, 1, 2, \ldots$ and inequality (2.5) in Lemma 1, under the assumptions $(A_1)_1-$ $(A_3)_1$, the right-hand side $F(t)$ and the kernel $K(t,\tau)$ are continuously differentiable functions in $[0,T]$ and $[0,T] \times [0,T]$, respectively. In addition, according to the assumptions $(A_1)_2 - (A_3)_2$ and formula (2.1), the conditions $F(t) > 0$ and $K(t,\tau) \geq 0$ are satisfied in $[0,T]$ and $[0,T] \times [0,T]$, respectively.

In addition, the solution of (4.2) is given by the series

$$q(t) = \sum_{n=0}^{\infty} (\mathbf{K}^n F)(t),$$

where $(\mathbf{K}F)(t) \equiv \int_0^t K(t,\tau)F(\tau)d\tau$. It is easy to verify that

$$|(\mathbf{K}^n F)(t)| \leq \|F\|_{\mathbf{C}[0,T]} \frac{\left(t \|K\|_{\mathbf{C}([0,T]\times[0,T])}\right)^n}{n!}, \quad t \in [0,T], \; n = 0, 1, \ldots.$$

Thus, we obtain the estimate

$$\|q\|_{\mathbf{C}[0,T]} \leq \|F\|_{\mathbf{C}[0,T]} \, e^{T\|K\|_{\mathbf{C}([0,T]\times[0,T])}}. \tag{4.5}$$

We therefore obtain a unique positive function $q(t)$, continuously differentiable in $[0,T]$, which the function (4.4) together with the solution of the problem (1.1)-(1.3) given by the Fourier series (4.1), form the unique solution of the inverse problem (1.1)-(1.4). Theorem 3 has been proved. □

The following result on continuously dependence on the data of the solution of the inverse problem (1.1)-(1.4) holds.

Theorem 4. *(Continuous dependence upon the data) Let $\Im$ be the class of triples in the form of $T = \{f, \varphi, E\}$ which satisfy the assumptions $(A_1)-(A_3)$ of Theorem 3 and*

$$\|f\|_{\mathbf{C}^{3,0}(\overline{D}_T)} \leq N_0, \|\varphi\|_{\mathbf{C}^3[0,1]} \leq N_1, \|E\|_{\mathbf{C}^1[0,T]} \leq N_2, 0 < N_3 \leq \min_{t\in[0,T]} |E(t)|,$$

for some positive constants N_i, $i = 0,\, 1,\, 2,\, 3$.

Then the solution pair (u, p) of the inverse problem (1.1)-(1.4) depends continuously upon the data in $\Im$.

Proof. Let us denote $\|T\| = (\|E\|_{\mathbf{C}^1[0,T]} + \|\varphi\|_{\mathbf{C}^3[0,1]} + \|f\|_{\mathbf{C}^{3,0}(\overline{D}_T)})$.

Let $T = \{f,\, \varphi,\, E\}$, $\bar{T} = \{\bar{f},\, \bar{\varphi},\, \bar{E}\} \in \Im$ be two sets of data. Let (p, u) and $(\bar{p}, \bar{u})$ be solutions of inverse problems (1.1)-(1.4) corresponding to the data Φ and $\bar{\Phi}$, respectively. Denote by $q(t) = e^{\int_0^t p(s)ds}$, $\bar{q}(t) = e^{\int_0^t \bar{p}(s)ds}$.

According to (4.1)and (4.2) we get:

$$
\begin{aligned}
q(t) &= F(t) + \int_0^t K(t,\tau) q(\tau) d\tau, \\
F(t) &= \frac{1}{E(t)} \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\varphi_n e^{-\lambda_n t} \int_0^1 y_n(x) dx \right], \\
K(t,\tau) &= \frac{1}{E(t)} \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[f_n(\tau) e^{-\lambda_n (t-\tau)} \int_0^1 y_n(x) dx \right],
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
\bar{q}(t) &= \bar{F}(t) + \int_0^t \bar{K}(t,\tau) \bar{q}(\tau) d\tau, \\
\bar{F}(t) &= \frac{1}{\bar{E}(t)} \sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\bar{\varphi}_n e^{-\lambda_n t} \int_0^1 y_n(x) dx \right], \\
\bar{K}(t,\tau) &= \frac{1}{\bar{E}(t)} \left(\sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} \left[\bar{f}_n(\tau) e^{-\lambda_n (t-\tau)} \int_0^1 y_n(x) dx \right] \right),
\end{aligned}
\tag{4.7}
$$

Taking into account the inequality in Corollary 1 the next inequalities will be true:

$$
\begin{aligned}
|F(t)| &\leq \frac{M}{|E(t)|} \left(\sum_{\substack{n=0 \\ (n \neq n_0)}}^{\infty} |(\varphi, u_n)| \right) \leq \\
&\leq \frac{M}{N_3} \|\varphi\|_{\mathbf{C}^3[0,1]} \leq \frac{M}{N_3} C N_1, \\
\left|\bar{K}(t,\tau)\right| &\leq \frac{M}{N_3} \max_{t \in [0,T]} \left\|\bar{f}(\cdot, t)\right\|_{\mathbf{C}^3[0,1]} \leq \frac{M}{N_3} C N_0,
\end{aligned}
\tag{4.8}
$$

where C is constant mentioned in Lemma 1, M is the constant which $0 < \int_0^1 y_n(x)dx \leq M$.

First let us estimate the difference $q-\bar{q}$. From (4.6) and (4.7) we obtain:
(4.9)

$$q(t)-\bar{q}(t) = F(t)-\bar{F}(t)+\int_0^t \left[K(t,\tau)-\bar{K}(t,\tau)\right] q(\tau)d\tau+\int_0^t \bar{K}(t,\tau)\left[q(\tau)-\bar{q}(\tau)\right]d\tau$$

Let $\alpha = \left\|F-\bar{F}\right\|_{\mathbf{C}[0,T]}+T\left\|K-K\right\|_{\mathbf{C}([0,T]\times[0,T])}\left\|q\right\|_{\mathbf{C}[0,T]}$.

Then denoting $Q(t)=|q(t)-\bar{q}(t)|$, equation (4.9) implies the inequality

$$Q(t) \leq \alpha+\int_0^t \left|\bar{K}(t,\tau)\right| Q(t)dt. \tag{4.10}$$

Applying the Gronwall inequality ([21, p. 9]), from (4.10) we obtain that

$$Q(t) \leq \alpha e^{\int_0^t \sup_{s\in[\tau,t]}\left|\bar{K}(s,\tau)\right|Q(t)dt}.$$

Finally using (4.5) and (4.8), we obtain

$$\left\|q-\bar{q}\right\|_{\mathbf{C}[0,T]} \leq C_1(\left\|F-\bar{F}\right\|_{\mathbf{C}[0,T]}+C_2\left\|K-\bar{K}\right\|_{\mathbf{C}([0,T]\times[0,T])}),$$

where $C_1 = e^{\frac{M}{N_3}CN_0T}$, $C_2 = \frac{MT}{N_3}CN_1C_1$. It will be seen from (4.8) that q continuously dependents upon F and K.

Let us show that F and K continuously dependents upon the data. It is easy to compute, with helping Corollary 1, that

$$\begin{aligned}
\left|\bar{F}(t)-F(t)\right| &= \left|\sum_{\substack{n=0\\(n\neq n_0)}}^{\infty}\left[\frac{\bar{\varphi}_n}{\bar{E}(t)}-\frac{\varphi_n}{E(t)}\right]e^{-\lambda_n t}\int_0^1 y_n(x)dx\right| \leq M\sum_{\substack{n=0\\(n\neq n_0)}}^{\infty}\left|(\frac{\bar{\varphi}}{\bar{E}(t)}-\frac{\varphi}{E(t)},u_n)\right| \\
&\leq \frac{M\bar{C}N_1}{N_3^2}\left\|E-\bar{E}\right\|_{\mathbf{C}[0,T]}+\frac{M\bar{C}}{N_3}\left\|\varphi-\bar{\varphi}\right\|_{\mathbf{C}^3[0,1]}
\end{aligned}$$

$$\begin{aligned}
\left|\bar{K}(t,\tau)-K(t,\tau)\right| &\leq M\sum_{\substack{n=0\\(n\neq n_0)}}^{\infty}\left|\frac{\bar{f}_n(\tau)}{\bar{E}(t)}-\frac{f_n(\tau)}{E(t)}\right| \\
&\leq \frac{M\bar{C}N_0}{N_3^2}\left\|E-\bar{E}\right\|_{\mathbf{C}[0,T]}+\frac{M\bar{C}}{N_3}\left\|f-\bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)},
\end{aligned}$$

By using last inequalities we obtain:

$$\begin{aligned}
\left\|F-\bar{F}\right\|_{\mathbf{C}[0,T]} &\leq M_1(\left\|E-\bar{E}\right\|_{\mathbf{C}[0,T]}+\left\|\varphi-\bar{\varphi}\right\|_{\mathbf{C}^3[0,1]}+\left\|f-\bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)}) \leq M_1\left\|T-\bar{T}\right\| \\
\left\|K-\bar{K}\right\|_{\mathbf{C}([0,T]\times[0,T])} &\leq M_2\left\|T-\bar{T}\right\|
\end{aligned}$$

where M_1 and M_2 are constants that are determined by constants N_k, $k=0,...,3$, M and $\bar{C}$. This means that F and K continuously dependent upon the data. Thus, q also continuously dependents upon the data, by (4.3).

Now, let us show that q' also depends continuously upon the data. Differentiating (4.6) and (4.7) with respect to t, we can obtain following representation:

$$\begin{aligned} q'(t) &= F'(t) + K(t,t)q(t) + \int_0^t K_t(t,\tau)q(\tau)d\tau, \\ \bar{q}'(t) &= \bar{F}'(t) + \bar{K}(t,t)\bar{q}(t) + \int_0^t \bar{K}_t(t,\tau)\bar{q}(\tau)d\tau. \end{aligned}$$

The following estimation holds:

$$\begin{aligned} \|q' - \bar{q}'\|_{\mathbf{C}[0,T]} \leq{} & \left\|F' - \bar{F}'\right\|_{\mathbf{C}[0,T]} \\ & + \left(\left\|K - \bar{K}\right\|_{\mathbf{C}([0,T]\times[0,T])} + T\left\|K_t - \bar{K}_t\right\|_{\mathbf{C}([0,T]\times[0,T])}\right)\|q\|_{\mathbf{C}[0,T]} \\ & + \left(\left\|\bar{K}\right\|_{\mathbf{C}([0,T]\times[0,T])} + T\left\|\bar{K}_t\right\|_{\mathbf{C}([0,T]\times[0,T])}\right)\|q - \bar{q}\|_{\mathbf{C}[0,T]}, \end{aligned} \tag{4.11}$$

Taking into account the facts that q, F and K continuously dependent upon the data, using the inequality (4.5) and inequality

$$\left|\bar{K}_t(t,\tau)\right| \leq \frac{N_2}{N_3^2} C\left\|\bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)} + \frac{C}{N_3}\left\|\bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)} \leq \left(\frac{N_2}{N_3^2} + \frac{1}{N_3}\right) C N_0,$$

it will be seen that q' depends continuously upon the F' and K_t. By using (2.5), we can obtain similar estimations for $\left\|F' - \bar{F}'\right\|_{\mathbf{C}[0,T]}$ and $\left\|K_t - \bar{K}_t\right\|_{\mathbf{C}([0,T]\times[0,T])}$, as

$$\begin{aligned} \left\|F' - \bar{F}'\right\|_{\mathbf{C}[0,T]} &\leq \frac{2MCN_1}{N_3^2}\left\|E - \bar{E}\right\|_{\mathbf{C}^1[0,T]} + \frac{MC}{N_3}\|\varphi - \bar{\varphi}\|_{\mathbf{C}^3[0,1]}, \\ \left\|K_t - \bar{K}_t\right\|_{\mathbf{C}([0,T]\times[0,T])} &\leq \frac{2MCN_0}{N_3^2}\left\|E - \bar{E}\right\|_{\mathbf{C}^1[0,T]} + \frac{MC}{N_3}\left\|f - \bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)}. \end{aligned}$$

Thus, using these inequalities from (4.11) we obtain

$$\|q' - \bar{q}'\|_{\mathbf{C}[0,T]} \leq M_3(\left\|E - \bar{E}\right\|_{\mathbf{C}^1[0,T]} + \|\varphi - \bar{\varphi}\|_{\mathbf{C}^3[0,1]} + \left\|f - \bar{f}\right\|_{\mathbf{C}^{3,0}(\overline{D}_T)}) \leq M_3\left\|T - \bar{T}\right\|,$$

where M_3 is constant that is determined by constants N_k, $k = 0, ..., 3$, M and C. It means that q' depends continuously upon the data as well.

The equality $p(t) = \frac{q'(t)}{q(t)}$ implies continuously dependence of p upon the data. Using the similar what we demonstrated above we can prove that u, which is given in (4.1), depends continuously upon the data. Theorem 4 has been proved. □

5. Conclusion

In this paper, we consider the initial-boundary value problem for the heat equation with a dinamic type boundary condition which is observed in the process of cooling of a thin solid bar one end of which is placed in contact in the case of perfect thermal contact and in a boundary reaction in diffusion of chemical. The Fourier method on the eigenfunctions of an auxiliary spectral problem with the boundary condition which is dependent on spectral parameter, is suitable for studying the problem under consideration. Under some regularity, consistency and orthogonality conditions, the existence, uniqueness and continuously dependence upon the data of the classical solution are shown. This paper also investigates the inverse problem of finding a coefficient of the heat equation from integral overdetermination condition's data.

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