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An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions

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A B S T R A C T

This paper investigates the inverse problem of finding a time-dependent coefficient in a heat equation with nonlocal boundary and integral overdetermination conditions. Under some regularity and consistency conditions on the input data, the existence, uniqueness and continuous dependence upon the data of the solution are shown by using the generalized Fourier method.

1. Introduction

Let $T > 0$ be a fixed number and $D_T = \{(x, t) : 0 < x < 1 : 0 < t \leq T\}$.

Consider the inverse problem of finding a pair of functions $\{p(t), u(x, t)\}$ such that it satisfies the equation

$$u_t = u_{xx} - p(t)u + f(x, t), \quad (x, t) \in D_T,$$

the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1,$$

the nonlocal boundary conditions

$$\alpha_1 u_x(0, t) + \beta_1 u(0, t) + \gamma_1 u(1, t) = 0, \quad 0 \leq t \leq T,$$
$$\alpha_2 u_x(1, t) + \beta_2 u(0, t) + \gamma_2 u(1, t) = 0, \quad 0 \leq t \leq T,$$

and the overdetermination condition

$$\int_0^1 u(x, t)dx = E(t), \quad 0 \leq t \leq T$$

where f, φ, E are given functions and $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ are given numbers with $rank \begin{bmatrix} \alpha_1 & 0 & \beta_1 & \gamma_1 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} = 2$.

This problem can be used in a heat transfer process where a source parameter is present. If we let $u(x, t)$ represent the temperature distribution, then the above-mentioned problem can be regarded as a control problem with a source control.

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The source control parameter $p(t)$ needs to be determined by thermal energy $E(t)$. The interested readers can refer to [1–3] for some examples.

In the cases $\alpha_1 \neq 0, \alpha_2 \neq 0$ and $\alpha_1 = 0, \alpha_2 = 0$, the above-mentioned inverse problem is studied in [4] by using an iterative method combined with a fundamental solution of the heat equation. This method is applicable when the coefficients $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ are time dependent but not applicable in the case where only one of the coefficients of α_1 and α_2 is different than zero. The present paper is devoted to the study of the inverse problem for the case $\alpha_1 \neq 0, \alpha_2 = 0$. Notice that this condition does not lose its generality because the other case is reduced to this one by changing variables x by $1 - x$.

Since the function p is space independent, $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ are constants and the boundary conditions are linear and homogeneous, the method of separation of variables is suitable for studying the problem which is mentioned above. The main difficulty in applying the Fourier method is its basisness, i.e. expansion in terms of eigenfunctions of an auxiliary spectral problem. A simple type of expansion exists for the problem with a self-adjoint linear differential expression and self-adjoint boundary conditions: by reason of the Hilbert–Schmidt expansion theorem in the theory of integral equations, any functions satisfying the self-adjoint boundary conditions can be expanded in a uniformly convergent, generalized Fourier series in terms of eigenfunctions of this problem (see [5]). A more difficult expansion theorem in terms of eigenfunctions exists for the problem with regular boundary conditions (see [6]): any functions satisfying the regular boundary conditions can be expanded in a uniformly convergent series in terms of eigenfunctions of this problem, when all eigenvalues of this problem are the simple zeros of the characteristic determinant. More informations can be found in [6].

The auxiliary spectral problem for the above-mentioned problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 \le x \le 1, \\ \alpha_1 X'(0) + \beta_1 X(0) + \gamma_1 X(1) = 0, \\ \alpha_2 X'(1) + \beta_2 X(0) + \gamma_2 X(1) = 0. \end{cases}$$

In the case $\alpha_1 \neq 0, \alpha_2 = 0$, the boundary conditions are regular iff $\gamma_2 \neq 0$. Without loss of generality, it can be assumed that $\alpha_1 = 1, \alpha_2 = 0, \gamma_2 = -1$, and $\gamma_1 = 0$. Therefore, the boundary conditions are given by $X'(0) + \alpha X(0) = 0, X(1) + \beta X(0) = 0$ where α and β are some constants. Notice that, the last boundary conditions are strongly regular when $\beta^2 \neq 1$ and not strongly regular when $\beta = \pm 1$. In general, the case of strongly regular boundary conditions is more comfortable in basisness point of view. In this paper, our aim is the investigation of the inverse problem in the case when the boundary conditions of the auxiliary spectral problem are not strongly regular, i.e. when $\beta = \pm 1$. We will study the inverse problem for the case $\beta = -1$. For the case $\beta = 1$, the problem can be analogously studied.

Briefly, we consider the next inverse problem with a regular boundary condition:

$$u_t = u_{xx} - p(t)u + f(x, t), \quad (x, t) \in D_T, \tag{1.1}$$

$$u(x, 0) = \varphi(x), \quad 0 \le x \le 1, \tag{1.2}$$

$$u_x(0, t) + \alpha u(0, t) = 0, \qquad u(0, t) - u(1, t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

$$\int_0^1 u(x, t)dx = E(t), \quad 0 \le t \le T. \tag{1.4}$$

Unlike the case of $\alpha \neq 0$, for the case $\alpha = 0$, the eigenvalues of the auxiliary spectral problem are the zeros of multiplicity two of the characteristic determinant. Since the expansion theorem in terms of eigenfunctions for the problem with regular boundary conditions is not applicable to the case $\alpha = 0$, it must be studied separately. In this case, the basisness of the eigenfunctions together with associated eigenfunctions are shown in [3] and the inverse problems for this case are studied in [7,8] by using the generalized Fourier method.

The problem of finding a coefficient $p(t)$ together with the solution $u(x, t)$ of the heat equation (1.1) with the integral overdetermination condition (1.4) and different nonlocal boundary conditions are studied in [9,10]. In [10,11], the inverse problem of finding the coefficient $p(t)$ from integral overdetermination data is also studied numerically. The inverse problems of determining a time-dependent coefficient in the heat equation with different boundary and overdetermination conditions can be referred in [12–14] and references therein.

Nonlocal boundary conditions like (1.4) arise from many important applications in heat transfer, thermoelasticity, control theory, life sciences, etc. For example, for heat propagation in a thin rod in which the law of variation $E(t)$ of the total quantity of heat in the rod is given in [3]. In [15], the nature of (1.3)-type boundary conditions is demonstrated.

The inverse problems of determining the coefficients in the heat equation have been investigated in many studies for the cases when the unknown coefficient is space-dependent in [16–21] and both time and space dependent in [22,23], to name only a few references.

The paper is organized as follows. In Section 1, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced. In Section 2, the existence and uniqueness of the solution of the inverse problem (1.1)–(1.4) is proved. Finally, the continuous dependence upon the data of the solution of the inverse problem is shown in Section 3.

2. The auxiliary spectral problem and some of its properties

Consider the spectral problem

$$\begin{cases} X''(x)+\lambda X(x)=0, & 0\le x\le 1,\\ X'(0)+\alpha X(0)=0, & X(0)=X(1), \end{cases} \tag{2.1}$$

with $\alpha \neq 0$. The problem (1.1) has the eigenvalues $\lambda_k,\ k=0,1,2,\ldots$ such that

$$\lambda_{2n}=(2\pi n)^2,\quad n=1,2,\ldots,$$
$$\lambda_{2n-1}=\mu_n^2,\quad n=1,2,\ldots,$$
$$\lambda_0=\begin{cases}\mu_0^2, & \alpha<0,\\ -s_0^2, & \alpha>0,\end{cases}$$

where $\mu_n=2\pi n+O(1)\in(2\pi n,2\pi n+\pi),\ n=0,1,2,\ldots$ and $\mu_n=2\pi n+O(1)\in(2\pi n-\pi,2\pi n),\ n=1,2,\ldots$ are monotone increasing positive solutions of the equation $\mu\sin\frac{\mu}{2}+\alpha\cos\frac{\mu}{2}=0$ in $\alpha<0$ and $\alpha>0$, respectively; s_0 is the unique positive solution of the equation $e^s=1+\frac{2\alpha}{s-\alpha}$ with $\alpha>0$. In addition, the system of eigenfunctions $X_k(x),\ k=0,1,2,\ldots$ is given by

$$\begin{aligned} &X_{2n}(x)=\cos(2\pi nx)-\frac{\alpha}{2\pi n}\sin(2\pi nx),\\ &X_{2n-1}(x)=\cos(\mu_n x)-\frac{\alpha}{\mu_n}\sin(\mu_n x),\quad n=1,2,\ldots,\\ &X_0(x)=\begin{cases}\cos\mu_0 x-\dfrac{\alpha}{\mu_0}\sin\mu_0 x, & \alpha<0,\\ \dfrac{s_0-\alpha}{s_0+\alpha}e^{s_0x}+e^{-s_0x}, & \alpha>0.\end{cases}\end{aligned} \tag{2.2}$$

Any functions satisfying the boundary conditions in (2.1) can be expanded in a uniformly convergent series in terms of eigenfunctions (2.2) by Theorem 5.3 in [6]. The sequence $X_n(x),\ n=0,1,\ldots$, is also a basis with parenthesis in $L_2[0,1]$ (see [24,25]).

The adjoint problem of (2.1) is in the form of

$$\begin{cases} Y''(x)+\lambda Y(x)=0, & 0\le x\le 1,\\ Y(1)=0, & Y'(1)-Y'(0)-\alpha Y(0)=0. \end{cases} \tag{2.3}$$

The eigenvalues of this problem are same as in the problem (2.1). The system of eigenfunctions $Y_n(x)$, $n=0,1,2,\ldots$ of the problem (2.1) is given by

$$\begin{aligned} &Y_{2n}(x)=-\frac{4\pi n}{\alpha}\sin(2\pi nx),\\ &Y_{2n-1}(x)=\frac{2\mu_n}{\alpha(\mu_n^2+\alpha^2-2\alpha)}((\mu_n^2-\alpha^2)\sin(\mu_n x)+2\alpha\mu_n\cos(\mu_n x)),\quad n=1,2,\ldots\\ &Y_0(x)=\begin{cases}\dfrac{2\mu_0}{\alpha(\mu_0^2+\alpha^2-2\alpha)}((\mu_0^2-\alpha^2)\sin(\mu_0 x)+2\alpha\mu_0\cos(\mu_0 x)), & \alpha<0,\\ -\dfrac{(s_0+\alpha)}{2\alpha(s_0^2-\alpha^2+2\alpha)}((s_0-\alpha)^2e^{s_0x}-(s_0+\alpha)^2e^{-s_0x}), & \alpha>0.\end{cases}\end{aligned} \tag{2.4}$$

Lemma 1. *The systems (2.2) and (2.4) form a biorthogonal system of functions on* $[0,1]$, *i.e. for all nonnegative integers i and j,*

$$(X_i,Y_j)=\int_0^1 X_i(x)Y_j(x)dx=\delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Proof. The proof of the fact that $(X_i,Y_j)=0$ for $i\neq j$ follows from the general theory of linear differential operators (see Theorem 2.2 in [6]). The equalities $(X_{2n},Y_{2n})=1$, $n=1,2,\ldots$ are immediately shown.

It is easy to show that

$$\begin{aligned}(X_{2n-1},Y_{2n-1}) = &\ \frac{1}{\alpha(\mu_n^2+\alpha^2-2\alpha)}\\ &\times\int_0^1[(2\mu_n\cos(\mu_n x)-2\alpha\sin(\mu_n x))((\mu_n^2-\alpha^2)\sin(\mu_n x)+2\alpha\mu_n\cos(\mu_n x))]dx\end{aligned}$$

$$= \frac{1}{\alpha(\mu_n^2+\alpha^2-2\alpha)}\left((\mu_n^2-3\alpha^2)\sin^2\mu_n+\alpha(\mu_n^2+\alpha^2)+\frac{\alpha(3\mu_n^2-\alpha^2)}{2\mu_n}\sin(2\mu_n)\right)$$
$$= 1, \quad n=1,2,\ldots$$

since μ_n is the solution of the equation $\mu\sin\frac{\mu}{2}+\alpha\cos\frac{\mu}{2}=0$ which is equivalent to $\sin\mu=-\frac{2\alpha\mu}{\mu^2+\alpha^2}$ and $\cos\mu=\frac{\mu^2-\alpha^2}{\mu^2+\alpha^2}$. In addition, $\mu_n^2+\alpha^2-2\alpha\neq 0$, $n=1,2,\ldots$ hold, because in the contrary case we take $\sin\mu_n=-\frac{2\alpha\mu_n}{\mu_n^2+\alpha^2}=-\mu_n$. However, the equation $\sin\mu=-\mu$ has not a positive solution. The equality $(X_0,Y_0)=1$ is analogously proved. □

Let $e_n(x)=\sin(\mu_n x)$ and $h_n(x)=\cos(\mu_n x)$, $n=1,2,\ldots$ be two sequences.

Lemma 2 (*Bessel-Type Inequalities*). *If $\psi(x)\in L_2[0,1]$, then the estimates*

$$\sum_{n=1}^{\infty}|(\psi,e_n)|^2\le c_1\|\psi\|^2_{L_2[0,1]} \quad \text{and} \quad \sum_{n=1}^{\infty}|(\psi,h_n)|^2\le c_2\|\psi\|^2_{L_2[0,1]} \quad (c_1 \text{ and } c_2 \text{ are constants})$$

hold where $(\psi,e_n)=\int_0^1\psi(x)e_n(x)dx$.

Proof. Let us prove the first inequality. It is known that $\mu_n=2\pi n+\delta_n$ where δ_n $(n=1,2,\ldots)$ is a bounded sequence. Therefore,

$$e_n(x)=\sin(\mu_n x)=\sin(2\pi nx)\cos(\delta_n x)+\cos(2\pi nx)\sin(\delta_n x).$$

Then

$$|(\psi,e_n)|^2\le 2\left(\int_0^1\psi(x)\sin(2\pi nx)\cos(\delta_n x)dx\right)^2+2\left(\int_0^1\psi(x)\cos(2\pi nx)\sin(\delta_n x)dx\right)^2.$$

The following estimate holds for the first sum of the last inequality by integrating by parts and using the Schwarz inequality:

$$\begin{aligned}\left(\int_0^1\psi(x)\sin(2\pi nx)\cos(\delta_n x)dx\right)^2 &= \left[\int_0^1\cos(\delta_n x)d\left(\int_0^x\psi(t)\sin(2\pi nt)\right)\right]^2\\
&= \left[-\cos(\delta_n)\int_0^1\psi(x)\sin(2\pi nx)dx\right.\\
&\quad \left.+\ \delta_n\int_0^1\sin(\delta_n x)\left(\int_0^x\psi(t)\sin(2\pi nt)dt\right)dx\right]^2\\
&\le const\left[\left|\int_0^1\psi(x)\sin(2\pi nx)dx\right|^2+\left(\int_0^1\left|\int_0^x\psi(t)\sin(2\pi nt)dt\right|dx\right)^2\right]\\
&\le const\left[\left|\int_0^1\psi(x)\sin(2\pi nx)dx\right|^2+\int_0^1\left|\int_0^x\psi(t)\sin(2\pi nt)dt\right|^2dx\right].\end{aligned}$$

Applying the Bessel inequality we obtain that

$$\sum_{n=1}^{\infty}\left|\int_0^1\psi(x)\sin(2\pi nx)dx\right|^2\le const\|\psi\|^2_{L_2[0,1]}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\int_0^1\left|\int_0^x\psi(t)\sin(2\pi nt)dt\right|^2dx &\le \int_0^1\sum_{n=1}^{\infty}\left|\int_0^x\psi(t)\sin(2\pi nt)dt\right|^2dx\\
&\le const\int_0^1\int_0^x\psi^2(t)dtdx\le const\|\psi\|^2_{L_2[0,1]}.\end{aligned}$$

Thus, we get

$$\sum_{n=1}^{\infty}\left(\int_0^1\psi(x)\sin(2\pi nx)\cos(\delta_n x)dx\right)^2\le const\|\psi\|^2_{L_2[0,1]}.$$

Similarly, we can prove that

$$\sum_{n=1}^{\infty}\left(\int_0^1 \psi(x)\cos(2\pi nx)\sin(\delta_n x)dx\right)^2 \le const\|\psi\|^2_{L_2[0,1]}.$$

The last two inequalities imply $\sum_{n=1}^{\infty}|(\psi, e_n)|^2 \le c_1\|\psi\|^2_{L_2[0,1]}$. The second inequality is proved similarly. □

Since the eigenfunction $Y_{2n-1}(x)$ consists of the functions $e_n(x)$ and $h_n(x)$, as $Y_{2n-1}(x) = a_n e_n(x) + b_n h_n(x)$ where $a_n = \frac{2\mu_n(\mu_n^2-\alpha^2)}{\alpha(\mu_n^2+\alpha^2-2\alpha)}$, $b_n = \frac{4\mu_n^2}{\mu_n^2+\alpha^2-2\alpha}$, the following corollary of Lemma 2 is obtained by using the Schwarz inequality.

Corollary 1. *If $\psi(x) \in L_2[0,1]$, then the following estimate holds:*

$$\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}|(\psi, Y_{2n-1})| \le c_3\|\psi\|_{L_2[0,1]} \quad (c_3 \text{ is constant}).$$

The following lemma is easily obtained by applying integration by parts twice.

Lemma 3. *If $\varphi(x) \in C^2[0,1]$ satisfies the conditions $\varphi'(0) - \alpha\varphi(0) = 0$, $\varphi(0) = \varphi(1)$ then the equalities*

$$(\varphi, Y_{2n-1}) = -\frac{1}{\mu_n^2}(\varphi'', Y_{2n-1}) \quad \text{and} \quad (\varphi, Y_{2n}) = -\frac{1}{(2\pi n)^2}(\varphi'', Y_{2n})$$

hold.

3. Existence and uniqueness of the solution of the inverse problem

The pair $\{p(t),\ u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T))$ for which the conditions (0.1)–(0.4) are satisfied, is called a classical solution of the inverse problem (0.1)–(0.4).

We have the following assumptions on φ, E and f:

(A_1)
- $(A_1)_1$ $\varphi(x) \in C^2[0,1]$; $\varphi'(0) - \alpha\varphi(0) = 0$, $\varphi(0) = \varphi(1)$;
- $(A_1)_2$ $\varphi_0 > 0$, $\varphi_{2n-1} \ge 0$, $n = 1, 2, \ldots$, when $\alpha < 0$; $\varphi_1 < 0$, $\varphi_{2n-1} \le 0$, $n = 2, 3, \ldots$, when $\alpha > 0$;

(A_2)
- $(A_2)_1$ $E(t) \in C^1[0,T]$; $E(0) = \int_0^1 \varphi(x)dx$;
- $(A_2)_3$ $E(t) > 0$, $\forall t \in [0,T]$;

(A_3)
- $(A_3)_1$ $f(x,t) \in C(\overline{D}_T)$; $f(x,t) \in C^2[0,1]$, $\forall t \in [0,T]$; $f_x(1,t) - \alpha f(0,t) = 0$, $f(0,t) = f(1,t)$;
- $(A_3)_3$ $f_0(\tau) \ge 0$, $f_{2n-1}(\tau) \ge 0$, $n = 1, 2, \ldots$, when $\alpha < 0$; $f_{2n-1}(\tau) \le 0$, $n = 2, 3, \ldots$, when $\alpha > 0$;

where $\varphi_n = \int_0^1 \varphi(x)Y_n(x)dx$, $f_n(t) = \int_0^1 f(x,t)Y_n(x)dx$, $n = 0, 1, 2, \ldots$.

The main result is presented as follows.

Theorem 1. *Let* (A_1)–(A_3) *be satisfied. Then, the inverse problem* (1.1)–(1.4) *has a unique classical solution.*

Proof. By applying the standard procedure of the Fourier method, we obtain the following representation of the solution of (0.1)–(0.3) for arbitrary $p(t) \in C[0,T]$:

$$u(x,t) = u_0(t)X_0(x) + \sum_{n=1}^{\infty}[u_{2n-1}(t)X_{2n-1}(x) + u_{2n}(t)X_{2n}(x)], \tag{3.1}$$

where

$$u_0(t) = \varphi_0 e^{-\lambda_0 t - \int_0^t p(s)ds} + \int_0^t f_0(\tau)e^{-\lambda_0(t-\tau) - \int_\tau^t p(s)ds}d\tau,$$

$$u_{2n}(t) = \varphi_{2n}e^{-(2\pi n)^2 t - \int_0^t p(s)ds} + \int_0^t f_{2n}(\tau)e^{-(2\pi n)^2(t-\tau) - \int_\tau^t p(s)ds}d\tau,$$

$$u_{2n-1}(t) = \varphi_{2n-1}e^{-\mu_n^2 t - \int_0^t p(s)ds} + \int_0^t f_{2n-1}(\tau)e^{-\mu_n^2(t-\tau) - \int_\tau^t p(s)ds}d\tau,$$

with $\varphi_n = \int_0^1 \varphi(x)Y_n(x)dx$, $f_n(t) = \int_0^1 f(x,t)Y_n(x)dx$, $k = 0, 1, 2, \ldots$.

Under the conditions $(A_1)_1$ and $(A_3)_1$ the series (3.1) and its x-partial derivative are uniformly convergent in $\overline{D}_T$ since their majorizing sums are absolutely convergent owing to Lemma 3. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{D}_T$. The t-partial derivative and the xx-second order partial derivative series are uniformly convergent for $t \ge \varepsilon > 0$ (ε is an arbitrary positive number). Thus, $u(x,t)$ is in class $C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ and satisfies the conditions (1.1)–(1.3) for arbitrary $p(t) \in C[0,T]$.

Applying the overdetermination condition (1.4), we obtain the following Volterra integral equation of the second kind with respect to $q(t) = e^{\int_0^t p(s)ds}$:

$$q(t) = F(t) + \int_0^t K(t,\tau)q(\tau)d\tau \tag{3.2}$$

where

$$\begin{aligned} F(t) &= \frac{1}{E(t)}\left(\varphi_0 e^{-\lambda_0 t}\int_0^1 X_0(x)dx + \sum_{n=1}^{\infty}\left[\varphi_{2n-1}\left(\frac{1}{\mu_n}\sin\mu_n - \frac{\alpha}{\mu_n^2}(1-\cos\mu_n)\right)e^{-\mu_n^2 t}\right]\right), \\ K(t,\tau) &= \frac{1}{E(t)}f_0(\tau)e^{-\lambda_0(t-\tau)}\int_0^1 X_0(x)dx \\ &\quad + \frac{1}{E(t)}\sum_{n=1}^{\infty}\left[f_{2n-1}(\tau)\left(\frac{1}{\mu_n}\sin\mu_n - \frac{\alpha}{\mu_n^2}(1-\cos\mu_n)\right)e^{-\mu_n^2(t-\tau)}\right]. \end{aligned} \tag{3.3}$$

It is easy to show that

$$\int_0^1 X_0(x)dx = \begin{cases} \dfrac{1}{\mu_0}\sin\mu_0 - \dfrac{\alpha}{\mu_0^2}(1-\cos\mu_0), & \alpha < 0 \\ 0, \quad \alpha > 0 \end{cases}$$

and

$$\frac{1}{\mu_n}\sin\mu_n - \frac{\alpha}{\mu_n^2}(1-\cos\mu_n) = -\frac{2\alpha}{\mu_n^2}, \quad n = 1, 2, \ldots.$$

In the case of the existence of the positive solution of (3.2) in class $C^1[0,T]$, the function $p(t)$ can be determined by $q(t) = e^{\int_0^t p(s)ds}$ such that

$$p(t) = \frac{q'(t)}{q(t)}. \tag{3.4}$$

Under the assumptions of $(A_1)_1$ and $(A_3)_1$ the right-hand side $F(t)$ and the kernel $K(t,\tau)$ are continuously differentiable functions in $[0,T]$ and $[0,T]\times[0,T]$, respectively by using the Lemma 3 and Corollary 1. In addition, according to the assumptions $(A_1)_2$–$(A_3)_2$, the conditions $F(t) > 0$ and $K(t,\tau) \geq 0$ are satisfied in $[0,T]$ and $[0,T]\times[0,T]$, respectively.

In addition, the solution of (3.2) is given by the series

$$q(t) = \sum_{n=0}^{\infty}(\mathbf{K}^n F)(t),$$

where $(\mathbf{K}F)(t) \equiv \int_0^t K(t,\tau)F(\tau)d\tau$. It is easy to verify that

$$|(\mathbf{K}^n F)(t)| \leq \|F\|_{C[0,T]}\frac{(t\|K\|_{C([0,T]\times[0,T])})^n}{n!}, \quad t\in[0,T],\ n = 0, 1, \ldots.$$

Thus, we obtain the estimate

$$\|q\|_{C[0,T]} \leq \|F\|_{C[0,T]}e^{T\|K\|_{C([0,T]\times[0,T])}}. \tag{3.5}$$

Therefore we obtain a unique positive function $q(t)$, continuously differentiable in $[0,T]$. The function (3.4) together with the solution of the problem (1.1)–(1.3) given by the Fourier series (3.1) form the unique solution of the inverse problem (1.1)–(1.4). Theorem 1 has been proved. □

4. Continuous dependence of the solution of the inverse problem upon the data

The following result for continuous dependence upon the data of the solution of the inverse problem (1.1)–(1.4) holds.

Theorem 2. *Let $\Im$ be the class of triples in the form of $\Phi = \{f, \varphi, E\}$ which satisfy the assumptions* (A_1)–(A_3) *of* Theorem 1 *and*

$$\|f\|_{C^{2,0}(\overline{D}_T)} \leq N_0, \qquad \|\varphi\|_{C^2[0,1]} \leq N_1, \qquad \|E\|_{C^1[0,T]} \leq N_2,$$

$$0 < N_3 \leq \min\left\{\min_{t\in[0,T]}|E(t)|,\ \min_{t\in[0,T]}|E'(t)|\right\},$$

for some positive constants $N_i,\ i = 0, 1, 2, 3$.

Then the solution pair (u, p) of the inverse problem (1.1)–(1.4) *depends continuously upon the data in $\Im$ for small N_0.*

Proof. Let us denote $\|\Phi\| = (\|E\|_{C^1[0,T]} + \|\varphi\|_{C^2[0,1]} + \|f\|_{C^{2,0}(\overline{D}_T)})$.

Let $\Phi = \{f, \varphi, E\}$, $\tilde{\Phi} = \{\tilde{f}, \tilde{\varphi}, \tilde{E}\} \in \Im$ be two sets of data. Let (p, u) and $(\tilde{p}, \tilde{u})$ be solutions of inverse problems (1.1)–(1.4) corresponding to the data Φ and $\tilde{\Phi}$, respectively. Denote by $q(t) = e^{\int_0^t p(s)ds}$, $\tilde{q}(t) = e^{\int_0^t \tilde{p}(s)ds}$.

According to (3.1) and (3.2) we get

$$q(t) = F(t) + \int_0^t K(t,\tau)q(\tau)d\tau, \tag{4.1}$$

$$F(t) = \frac{1}{E(t)}\left(\varphi_0 e^{-\lambda_0 t}\int_0^1 X_0(x)dx - \sum_{n=1}^{\infty}\frac{2\alpha}{\mu_n^2}\varphi_{2n-1}e^{-\mu_n^2 t}\right),$$

$$K(t,\tau) = \frac{1}{E(t)}\left(f_0(\tau)e^{-\lambda_0(t-\tau)}\int_0^1 X_0(x)dx - \sum_{n=1}^{\infty}\left[\frac{2\alpha}{\mu_n^2}f_{2n-1}(\tau)e^{-\mu_n^2(t-\tau)}\right]\right),$$

and

$$\tilde{q}(t) = \tilde{F}(t) + \int_0^t \tilde{K}(t,\tau)\tilde{q}(\tau)d\tau, \tag{4.2}$$

$$\tilde{F}(t) = \frac{1}{\tilde{E}(t)}\left(\tilde{\varphi}_0 e^{-\lambda_0 t}\int_0^1 X_0(x)dx - \sum_{n=1}^{\infty}\frac{2\alpha}{\mu_n^2}\tilde{\varphi}_{2n-1}e^{-\mu_n^2 t}\right),$$

$$\bar{K}(t,\tau) = \frac{1}{\tilde{E}(t)}\left(\tilde{f}_0(\tau)e^{-\lambda_0(t-\tau)}\int_0^1 X_0(x)dx - \sum_{n=1}^{\infty}\left[\frac{2\alpha}{\mu_n^2}\tilde{f}_{2n-1}(\tau)e^{-\mu_n^2(t-\tau)}\right]\right).$$

First, let us estimate the difference $q - \tilde{q}$. From (4.1) and (4.2) we obtain

$$\begin{aligned} q(t) - \tilde{q}(t) = {} & F(t) - \tilde{F}(t) + \int_0^t [K(t,\tau) - \tilde{K}(t,\tau)]q(\tau)d\tau + \int_0^t \tilde{K}(t,\tau)[q(\tau) - \tilde{q}(\tau)]d\tau \\ & \Rightarrow \|q - \tilde{q}\|_{C[0,T]} \le \|F - \tilde{F}\|_{C[0,T]} + T\|K - \tilde{K}\|_{C([0,T]\times[0,T])}\|q\|_{C[0,T]} \\ & \quad + T\|\tilde{K}\|_{C([0,T]\times[0,T])}\|q - \tilde{q}\|_{C[0,T]}. \end{aligned} \tag{4.3}$$

Taking into account the inequality in Corollary 1, the next inequalities will be true:

$$\begin{aligned} |F(t)| & \le \frac{1}{|E(t)|}\left(|(\varphi, Y_0)|\int_0^1 |X_0(x)|dx + 2|\alpha|\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}|(\varphi, Y_{2n-1})|\right) \\ & \le \frac{1}{N_3}(\|X_0\|_{L_2[0,1]}\|Y_0\|_{L_2[0,1]} + 2|\alpha|c)\|\varphi\|_{L_2[0,1]} \\ & \le \frac{c_4}{N_3}N_1, \quad (c_4 = \|Y_0\|_{L_2[0,1]}\|X_0\|_{L_2[0,1]} + 2|\alpha|c \text{ is constant}), \end{aligned} \tag{4.4}$$

$$|\tilde{K}(t,\tau)| \le \frac{c_5}{N_3}\max_{t\in[0,T]}\|\tilde{f}(\cdot,t)\|_{L_2[0,1]} \le \frac{c_4}{N_3}N_0.$$

It can be seen from (4.3) that q is continuously dependent upon F and K when $1 - \frac{c_4}{N_3}N_0 T > 0$. This condition is obtained for small N_0.

Let us show that F and K are continuously dependent upon the data. It is easy to compute, with the help of the Schwarz inequality and the inequality in Corollary 1, that

$$\left\|\frac{\varphi_0}{E} - \frac{\tilde{\varphi}_0}{\overline{E}}\right\|_{C[0,T]} \le M_1\|E - \tilde{E}\|_{C[0,T]} + M_2\|\varphi - \overline{\tilde{\varphi}}\|_{C[0,1]},$$

$$\begin{aligned} \left|\sum_{n=1}^{\infty}\frac{2\alpha}{\mu_n^2}\left[\frac{\tilde{\varphi}_{2n-1}}{\tilde{E}(t)} - \frac{\varphi_{2n-1}}{E(t)}\right]e^{-\mu_n^2 t}\right| & \le \sum_{n=1}^{\infty}2|\alpha|\frac{1}{\mu_n^2}\left|\left(\frac{\tilde{\varphi}}{\tilde{E}(t)} - \frac{\varphi}{E(t)}, Y_{2n-1}\right)\right| \\ & \le M_3\|E - \tilde{E}\|_{C[0,T]} + M_4\|\varphi - \tilde{\varphi}\|_{C[0,1]} \end{aligned}$$

$$\left\|\frac{f_0}{E} - \frac{\tilde{f}_0}{\overline{\overline{E}}}\right\|_{C[0,T]} \le M_5\|E - \tilde{E}\|_{C[0,T]} + M_6\|f - \tilde{f}\|_{C(\overline{D}_T)},$$

$$\left(\sum_{n=1}^{\infty}\left[\frac{2\alpha}{\mu_n^2}\left[\frac{\bar{f}_{2n-1}(\tau)}{\tilde{E}(t)} - \frac{f_{2n-1}(\tau)}{E(t)}\right]e^{-\mu_n^2(t-\tau)}\right]\right) \le M_7\|E - \tilde{E}\|_{C[0,T]} + M_8\|f - \bar{f}\|_{C(\overline{D}_T)},$$

where $M_k,\ k=1,\dots,8$ are constants that are determined by $N_k,\ k=1,\dots,4$. By using last inequalities we obtain

$$\|F-\tilde{F}\|_{C[0,T]}\le M_9(\|E-\tilde{E}\|_{C[0,T]}+\|\varphi-\tilde{\varphi}\|_{C[0,1]}+\|f-\tilde{f}\|_{C(\bar{D}_T)})\le M_9\|\Phi-\tilde{\Phi}\|.$$

This means that F and K are continuously dependent upon the data. Thus, q is also continuously dependent upon the data by (4.3).

Now, let us show that q' also depends continuously upon the data. Differentiating (4.1) and (4.2) with respect to t, we can obtain the following representations:

$$q'(t)=F'(t)+K(t,t)q(t)+\int_0^t K_t(t,\tau)q(\tau)d\tau,$$

$$\tilde{q}'(t)=\tilde{F}'(t)+\bar{K}(t,t)\tilde{q}(t)+\int_0^t \bar{K}_t(t,\tau)\tilde{q}(\tau)d\tau.$$

The following estimation holds:

$$\|q'-\tilde{q}'\|_{C[0,T]}\le\|F'-\tilde{F}'\|_{C[0,T]}+(\|K-\bar{K}\|_{C([0,T]\times[0,T])}+T\|K_t-\bar{K}_t\|_{C([0,T]\times[0,T])})\|q\|_{C[0,T]}$$
$$+(\|\bar{K}\|_{C([0,T]\times[0,T])}+T\|\bar{K}_t\|_{C([0,T]\times[0,T])})\|q-\tilde{q}\|_{C[0,T]}.$$

Taking into account the inequality (3.5), the inequality in (4.4) and the inequalities

$$|\bar{K}_t(t,\tau)|\le\frac{N_2}{N_3^2}c_5\|\tilde{f}\|_{C(\bar{D}_T)}+\frac{c_6}{N_3}\|\tilde{f}\|_{C^{2,0}(\bar{D}_T)}\le\left(\frac{N_2}{N_3^2}c_5+\frac{c_6}{N_3}\right)N_0,\quad (c_5\text{ and }c_6\text{ are constants}),$$

it will be seen that q' depends continuously upon the F' and K_t. By using Lemma 3 and Corollary 1, we can obtain similar estimations for $\|F'-\bar{F}'\|_{C[0,T]}$ and $\|K_t-\bar{K}_t\|_{C([0,T]\times[0,T])}$, as

$$\|F'-\tilde{F}\|_{C[0,T]}\le M_{10}\|E-\tilde{E}\|_{C^1[0,T]}+M_{11}\|\varphi''-\tilde{\varphi}''\|_{C[0,1]},$$

$$\|K_t-\bar{K}_t\|_{C([0,T]\times[0,T])}\le M_{12}\|E-\tilde{E}\|_{C^1[0,T]}+M_{13}\|f_{xx}-\tilde{f}_{xx}\|_{C(\bar{D}_T)}.$$

Thus,

$$\|q'-\tilde{q}'\|_{C[0,T]}\le M_{14}(\|E-\tilde{E}\|_{C^1[0,T]}+\|\varphi''-\tilde{\varphi}''\|_{C[0,1]}+\|f_{xx}-\tilde{f}_{xx}\|_{C(\bar{D}_T)})\le M_{14}\|\Phi-\tilde{\Phi}\|.$$

It means that q' depends continuously upon the data as well.

The equality $p(t)=\frac{q'(t)}{q(t)}$ implies the continuous dependence of p upon the data. Using the similar what we demonstrated above we can prove that u, which is given in (3.1), depends continuously upon the data. Theorem 2 has been proved. □

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