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ON THE BASISETY OF EXPONENTS IN L_p

Abstract

The paper gives an analogue of the famous theorem of "$\frac{1}{4}$ - Kadets" on the basis of perturbated systems an exponent in L_p.

Well-known classical theorem of Paley-Wiener [1] that if the system $d \equiv \sup_n |\lambda_n - n| < \frac{1}{\pi^2}$, then

$$\left\{ e^{i\lambda_n t} \right\}, \quad n \in Z; \tag{1}$$

forms a basis in $L_2(-\pi, \pi)$ being isomorphic to the classical system of exponentials $\left\{ e^{int} \right\}$, $n \in Z$ (Z is the set of integers), i.e. a Riesz basis, where $\{\lambda_n\} \subset R$ is some sequence of real numbers. In this work the question of improvement constant $\frac{1}{\pi^2}$ in the previous inequality. The final result belongs to M.I. Kadets [2], who proved that the same assertion has its place in the $d < \frac{1}{4}$, with the constant $\frac{1}{4}$ is non-improvement, i.e. $\exists \{\lambda_n\}$: $d \geq \frac{1}{4}$, for which the system (1) does not forms a basis in L_2.

In a different way, this fact can be interpreted in the following way. Let the Banach space of l_∞ sequences from the R with sup-norm: $\|\{a_n\}\|_\infty = \sup_n |a_n|$. Thus, the property of basis of $\left\{ e^{i\lambda_n t} \right\}_{n \in Z}$ in L_2 is stable in l_∞ on the sequence of $\{\delta_n\}_{n \in Z}$, where $\delta_n \equiv \lambda_n - n$. The question arises: Is this true, in $L_p \equiv L_p(-\pi, \pi)$, also i.e., is there a $c_p > 0$, so that when $d < c_p$ the system (1) forms a basis in L_p, isomorphic to $\left\{ e^{int} \right\}_{n \in Z}$, $I < p < +\infty$. By imposing various conditions on the sequence of $\{\delta_n\}_{n \in Z}$, more specifically, taking $\{\delta_n\}$ from the linear subspaces of some l_∞ in works [3-5] proved the validity of this fact in the L_p.

This article provides a synthesis of these results.

1. The necessary concepts and facts. First, we introduce the following class of sequences. Let $r > 0$ be a number. Let us denote by V_r class:

$$V_r \equiv \left\{ \{a_n\}_{n \in Z} : \|\{a_n\}\|_{V_r} < +\infty \right\},$$

where

$$\|\{a_n\}\|_{V_r} = \sup_{\{n_k\}} \left\{ \sum_{k=1}^{m} |a_{n_{k+1}} - a_{n_k}|^r \right\}^{1/r},$$

sup is taken over all increasing sequences of integers $\{ n_k\}_{k \in N} \subset Z$ (N-set of natural numbers). Easy to notice that the $V_r \subset l_\infty$.

Let $f \in L_1$. After $\{f_n\}_{n \in Z}$ we denote the Fourier coefficients of function f on system $\left\{ e^{int} \right\}_{n \in Z}$:

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \quad \forall n \in Z.$$

We say that $\{\delta_n\}_{n\in Z}$ is a multiplier of type (p,q), if for $\forall f\in L_p$, the element of sequence $\{\delta_n f_n\}_{n\in Z}$ is a Fourier coefficients of a function $g(t)$ of L_q, i.e., $g_n=\delta_n f_n$, $\forall n\in Z$. We will need the type of multipliers (p,p). $1<p<+\infty$. We know that $(2,2)\equiv l_\infty$ (see [6]). Here is a result of Hirschman on the multiplier type (p,p) (see [6]).

1) Let $\delta_n=\underline{\underline{O}}\left(|n|^{-\alpha}\right)$ where $n\to\infty$, for any $\alpha>0$ and $\{\delta_n\}_{n\in Z}\in V_r$ for any $r>2$, then $\{\delta_n\}_{n\in Z}\in(p,p)$, $\forall p\in\left(\frac{2r}{r+2},\frac{2r}{r-2}\right)$. If $\{\delta_n\}_{n\in Z}\in V_r$ where $1\le r<2$, then $\{\delta_n\}_{n\in Z}\in(p,p)$, $\forall p\in[1,\infty]$.

2) Let $\delta_n=\underline{\underline{O}}\left(|n|^{-\alpha}\right)$ where $n\to\infty$ and $\alpha\in\left(0,\frac{1}{2}\right]$. Then

$$\{\delta_n\}_{n\in Z}\in(p,p),\quad \forall p\in\left(\frac{2}{1+2\alpha},\frac{2}{1-2\alpha}\right).$$

Let us mention one more statement concerning the multipliers.

Proposition 1. If $\{\delta_n\}_{n\in Z}\in(p,q)$, $1\le p,q\le+\infty$, then $\exists\,\delta_{pq}>0$:

$$\left\|\sum\delta_n f_n\cdot e^{int}\right\|_q\le\delta_{pq}\left\|\sum f_n e^{int}\right\|_p, \tag{2}$$

for any finite sum $\sum$, where $\|\cdot\|_p-$ is the customary norm in L_p.

inf $\{\delta_{pq}$: satisfy the inequality (2)$\}$ is called the norm of multiplier $\{\delta_n\}_{n\in Z}$ and is denoted as $\|\{\delta_n\}\|$.

2. Main results. Let us contemplate the system (1), where $\lambda_n=n+\delta_n$, $\forall n\in Z$. It is true.

Theorem 1. *Let $\{\delta_n\}_{n\in Z}\in(p,p)$, $1<p<+\infty$. Then $\exists c_p>0$, and where $\|\{\delta_n\}\|<c_p$, the system (1) forms a basis in L_p, being isomorphic to $\left\{e^{int}\right\}_{n\in Z}$.*

Proof. Let $\sum\limits_n f_n\left(e^{i\lambda_n t}-e^{int}\right)$ be any finite sum. Taking into account the identity

$$e^{i\lambda_n t}-e^{int}=\left(e^{i\delta_n t}-1\right)e^{int}=\sum_{k=1}^{\infty}\frac{(i\delta_n t)^k}{k!}e^{int}=\sum_{k=1}^{\infty}\frac{(it)^k}{k!}\delta_n^k/e^{int},$$

we get:

$$\left\|\sum_n f_n\left(e^{i\lambda_n t}-e^{int}\right)\right\|_p=\left\|\sum_n f_n\sum_{k=1}^{\infty}\frac{(it)^k}{k!}\delta_n^k e^{int}\right\|_p=$$

$$=\left\|\sum_{k=1}^{\infty}\frac{(it)^k}{k!}\sum_n f_n\delta_n^k e^{int}\right\|_p\le\sum_{k=1}^{\infty}\frac{\pi^k}{k!}\left\|\sum_n\delta_n^k f_n e^{int}\right\|_p.$$

From the assertion (1) we directly get that

$$\left\|\sum_n\delta_n^k f_n e^{int}\right\|_p\le c_p^k\left\|\sum_n f_n e^{int}\right\|_p,$$

where $c_p=\delta_{pp}$.

As a result, we obtain

$$\left\|\sum_n f_n\left(e^{i\lambda_n t}-e^{int}\right)\right\|_p \le \left(e^{\pi c_p}-1\right)\left\|\sum_n f_n e^{int}\right\|_p . \qquad (3)$$

It is absolutely obvious that if $\{\delta_n\}\in(p,p)$, then $\{\delta\cdot\delta_n\}\in(p,p)$, for $\forall\,\delta\in R$. Consequently, there is norm of multiplier as small as is wished. Thus, in inequality (3) the constant c_p can be taken as small as is wished. Let's take $\forall\, f\in L_p$ and as $\{f_n\}$ in (3) we take the Fourier coefficients of functions f.

From (3) directly follows that the series $\sum\limits_{-\infty}^{+\infty} f_n\left(e^{i\lambda_n t}-e^{int}\right)$ in L_p matches. Let's consider the operator T :

$$Tf=\sum_{-\infty}^{+\infty} f_n\left(e^{i\lambda_n t}-e^{int}\right).$$

Then, we obtain from (3), that $\|Tf\|_p\le\left(e^{\pi c_p}-1\right)\|f\|_p$, and it means, that $\|T\|\le e^{\pi c_p}-1<1$. As a result of operator $I+T$, where $I:L_p\to L_p$ is a single operator. Easy to notice, that $(I+T)\left[e^{int}\right]=e^{i\lambda_n t},\ \forall n\in Z$. Thus we conclude the proof.

Where $p=2$ its easy to show that $\sup\limits_n|\delta_n|=\|\{\delta_n\}\|$, i.e. $\|\{\delta_n\}\|_{l_\infty}=\|\{\delta_n\}\|$, and moreover $(2.2)=l_\infty$. Therefore, this theorem can be considered L_p analogue to the theorem "$\frac{1}{4}$-Kadets".

Using the Hirschman's results, from this theorem we straightly get the following specific cases.

Corollary 1. *Let it be* $\delta_n=\delta\cdot\delta'_n,\ \ \forall n\in Z$;

1) $\delta'_n=\underline{\underline{O}}\left(|n|^{-\alpha}\right)$ *where* $n\to\infty$, *for any* $\alpha>0$ *and* $\left\{\delta'_n\right\}_{n\in Z}\in V_r,\quad r>2$. *Then* $\exists\, c_p>0$: *where* $\forall\,\delta\in[0,\,c_p)$, *the system* (1) *forms the basis in* L_p *being isomorphic to* $\left\{e^{int}\right\}_{n\in Z}$, $\forall p\in\left(\frac{2r}{r+2},\frac{2r}{r-2}\right)$. *Thus, if* $\left\{\delta'_n\right\}_{n\in Z}\in V_r,\ 1\le r<2$, *then the previous assertion obtains in* $\forall\,p\in(1,+\infty)$.

2) If $\delta'_n=\underline{\underline{O}}\left(|n|^{-\alpha}\right)$, *where* $n\to\infty$ *and* $\alpha\in\left(0,\frac{1}{2}\right)$, *then* $\exists c_p>0:\forall\delta\in[0,c_p)$ *conclusion of part 1) obtains in* $\forall\,p\in\left(\frac{2}{1+2\alpha},\frac{2}{1-2\alpha}\right)$.

Remark 1. Easy to show, that if $\{\delta_n\}\in(p,p)$, then $\left\{\tilde{\delta}_n\right\}$ is also belongs to the class of (p,p), if $card\left\{n:\ \delta_n\ne\tilde{\delta}_n\right\}<+\infty$

Let's examine the following example. Let it be

$$\delta_n=\begin{cases}\beta_1, & n\ge n_1\\ \beta_2, & n\le n_2\end{cases}$$

where $n_i\in Z,\quad i=1,2$; is any number. Based on Comment 1, first we examine the case of $n_1=0,\ n_2=-1$. Then from Riesz property (see eg. [7]) we get

$$\left\|\sum_{-N_1}^{0} a_n e^{int}\right\|_p+\left\|\sum_{1}^{N_2} a_n e^{int}\right\|_p\le M\left\|\sum_{-N_1}^{N_2} a_n e^{int}\right\|_p,\qquad 1<p<+\infty;$$

where $N_i \in Z$, $i = 1, 2$; M - depending only on constant p, it follows that $\{\delta_n\} \in (p,p)$, and $\|\{\delta_n\}\| = \max\{|\beta_1|;|\beta_2|\}$. Then from Theorem 1 we obtain that, $\exists c_p > 0$: where $\forall \beta_i \in (-c_p, c_p)$, $i = 1, 2$; the system (1) forms a basis of L_p, being isomorphic to $\{e^{int}\}_{n \in Z}$.

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