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On basicity of exponential systems in Sobolev-Morrey spaces

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Abstract *This work is devoted to the study of basicity of the system* $\{t\} \cup \{e^{int}\}_{n \in Z}$ *in one subspace of Sobolev-Morrey space.*

Keywords Sobolov-Morrey spaces · basicity

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1 Introduction

The concept of Morrey space was introduced by Morrey in 1938. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [5, 13]), this space also provides a large class of examples of mild solutions to the Navier–Stokes system [12]. In the context of fluid dynamics, Morrey spaces have been used to model flow when vorticity is a singular measure supported on certain sets in R^n [7]. There are sufficiently wide investigations related to fundamental problems in these spaces in view of differential equations, potential theory, maximal and singular operator theory, approximation theory and others (see, for example, [6] and the references above). More details about Morrey spaces can be found in [15, 17].

In recent years there has been a growing interest in the study of various subjects related to Morrey-type spaces. For example, some problems in harmonic analysis and approximation theory have been treated in [8–11, 16].

The basis properties of trigonometric systems in classical spaces are well studied. Study of the problems of the approximation theory in spaces such as Morrey has recently started and it remains much to learn. Basicity of exponential systems in Morrey-type spaces is studied in [3, 4]. In this paper we study the problem of basicity of exponential system in Sobolev-Morrey spaces. In the future, our goal is to follow the scheme of works [1,2,14].

2 Morrey-Lebesgue space

Let us give a definition for above-mentioned spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 \le \alpha \le 1$, $p \ge 1$, we mean a normed space of all functions $f(\xi)$ measurable on Γ equipped with a finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \left(\sup_B \left|B \bigcap \Gamma\right|^{\alpha-1} \int_{B\bigcap\Gamma} |f(\xi)|^p |d\xi|\right)^{1/p} < +\infty,$$

where the sup is taken over all disks B centered on Γ. $L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$.

The embedding $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \le \alpha_1 \le \alpha_2 \le 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_p(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \ge 1$. The case of $\Gamma = [-\pi,\pi]$ will be denoted by $L^{p,\alpha}$.

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ consisting of functions whose shifts are continuous in $L^{p,\alpha}$, i.e.

$$\tilde{L}^{p,\alpha} = \{f \in L_{p,\alpha} : \|f(+\delta) - f(\cdot)\| \to 0, \delta \to 0\}.$$

The closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ will be denoted by $ML^{p,\alpha}$, i.e. $ML^{p,\alpha} = \overline{\tilde{L}^{p,\alpha}}$.

3 Morrey-Sobolev space

Let $0 \le \alpha \le 1$, $p \ge 1$. By $W^1_{p,\alpha}$ we denote the space of functions which belong, together with their derivatives of first order, to the space $L^{p,\alpha}(\Gamma)$ equipped with the norm

$$\|f\|_{W^1_{p,\alpha}} = \|f\|_{L^{p,\alpha}} + \left\|f'\right\|_{L^{p,\alpha}}. \tag{3.1}$$

Denote by $\tilde{W}^1_{p,\alpha}$ the linear subspace of $W^1_{p,\alpha}$ consisting of functions whose first order derivatives are continuous with respect to the shift operator. By $MW^1_{p,\alpha}$ we denote the closure of this space with respect to the norm (3.1).

By $\mathscr{L}_{p,\alpha}$ we denote the direct sum of $ML^{p,a}$ and C (C is the complex plane)

$$\mathscr{L}_{p,\alpha} = ML^{p,\alpha} \oplus C.$$

Let us define the norm in $\mathscr{L}_{p,\alpha}$ in the following way:

$$\|\hat{u}\|_{\mathscr{L}_{p,\alpha}} = \|u\|_{L^{p,\alpha}} + |\lambda|, \forall \hat{u} = (u;\lambda) \in \mathscr{L}_{p,\alpha}.$$

The following lemma is true.

Lemma 1. *The operator* $(A\hat{u})(t) = \lambda + \int_{-\pi}^{t} u(\tau)\,d\tau$ *is an isomorphism from* $\mathscr{L}_{p,\alpha}$ *onto* $MW_{p,\alpha}^{1}$.

Proof. At first, let us show that $v(t) = (A\hat{u})\,t \in W_{p,\alpha}^{1}$. Indeed, since $L^{p,\alpha} \subset L_p \subset L_1$, then

$$\|v(t)\|_{L^{p,\alpha}} = \left\|\lambda + \int_{-\pi}^{t} u(\tau)\,d\tau\right\|_{L^{p,\alpha}} \leq \|\lambda\|_{L_{p,\alpha}} + \left\|\int_{-\pi}^{t} u(\tau)\,d\tau\right\|_{L^{p,\alpha}} \leq$$

$$\leq (2\pi)^{\frac{\alpha}{p}}|\lambda| + \sup_{I\subset(-\pi,\pi)}\left\{\frac{1}{|I|^{1-\alpha}}\int_I\left|\int_{-\pi}^{t} u(\tau)\,d\tau\right|^p dt\right\}^{1/p} \leq$$

$$\leq (2\pi)^{\frac{\alpha}{p}}|\lambda| + \sup_{I\subset(-\pi,\pi)}\left\{\frac{1}{|I|^{1-\alpha}}\int_I\left(\int_{-\pi}^{\pi} |u(\tau)|\,d\tau\right)^p dt\right\}^{1/p} =$$

$$= (2\pi)^{\frac{\alpha}{p}}|\lambda| + (2\pi)^{\frac{\alpha}{p}}\|u\|_{L_1(-\pi,\pi)} < +\infty. \tag{3.2}$$

Also, since $v'(t) = u(t) \in L^{p,\alpha}$, we have $v(t) \in W_{p,\alpha}^{1}$.

Now we show that $v(t) \in MW_{p,\alpha}^{1}$. From $u \in ML^{p,\alpha}$ it follows

$$\|v(\cdot+\delta) - v(\cdot)\|_{W_{p,\alpha}^{1}} = \|v(\cdot+\delta) - v(\cdot)\|_{L^{p,\alpha}} + \left\|v^{'}(\cdot+\delta) - v^{'}(\cdot)\right\|_{L^{p,\alpha}} =$$

$$= \left\|\int_{\cdot}^{\cdot+\delta} u(\tau)\,d\tau\right\|_{L^{p,\alpha}} + \|u(\cdot+\delta) - u(\cdot)\|_{L^{p,\alpha}} \to 0, \delta \to 0.$$

Let us show that A is a bounded operator. We have

$$\|A(\hat{u})\|_{W_{p,\alpha}^{1}} = \left\|\lambda + \int_{-\pi}^{t} u(\tau)\,d\tau\right\|_{L^{p,\alpha}} + \|u(\tau)\|_{L^{p,\alpha}}.$$

Taking into account (3.2)

$$\|A(\hat{u})\|_{W_{p,\alpha}^{1}} \leq (2\pi)^{\frac{\alpha}{p}}|\lambda| + (2\pi)^{\frac{\alpha}{p}}\|u\|_{L_1(-\pi,\pi)} + \|u\|_{L^{p,\alpha}}.$$

As the following relation holds

$$\|u\|_{L_1} \leq C_1\|u\|_{L_p} \leq C_2\|u\|_{L^{p,\alpha}},$$

we have the validity of the following inequality

$$\|A(\hat{u})\|_{W_{p,\alpha}^{1}} \leq M(|\lambda| + \|u\|_{L^{p,\alpha}}) = M\|\hat{u}\|_{\mathscr{L}_{p,\alpha}}, M = const.$$

Let us show that $\ker A = \{0\}$. Let $A\hat{u} = 0$, i.e. $\lambda + \int_{-\pi}^{t} u(\tau)\,d\tau = 0$. If we differentiate both sides, we get $u(t) = 0$, a.e. . Thus $\lambda = 0$. We have $\hat{u} = 0$. For $\forall v \in MW_{p,\alpha}^{1}$ taking $\hat{v} = (v'; v(-\pi))$ we have $\hat{v} \in L_{p,\alpha}$ and $A(\hat{v}) = v$. It means that $R_A = MW_{p,\alpha}^{1}$, where R_A is a range of the operator A . It follows from Banach's theorem on the inverse operator that the inverse of A is a continuous operator. The lemma is proved.

The following theorem is true.

Theorem 1. *System* $t\bigcup\left\{e^{int}\right\}_{n\in Z}$ *forms a basis for* $MW^1_{p,\alpha}\left(-\pi,\pi\right)$.

Proof. It is known that system $\left\{e^{int}\right\}_{n\in Z}$ is a basis in space $ML^{p,\alpha}$ [3]. Let us prove that the system $\{\hat{u}_{-1}\}\bigcup\{u_0\}\bigcup\{\hat{u}_n^{\pm}\}$,$n\geq 1$ forms a basis for $\mathscr{L}_{p,\alpha}$, where

$$\hat{u}_{-1}=\begin{pmatrix}1\\-\pi\end{pmatrix},\hat{u}_0=\begin{pmatrix}0\\1\end{pmatrix},\hat{u}_n^{+}=\begin{pmatrix}ine^{int}\\e^{-i\pi n}\end{pmatrix},\hat{u}_n^{-}=\begin{pmatrix}-ine^{-int}\\e^{i\pi n}\end{pmatrix},n\geq 1.$$

Let us show that for $\forall\hat{u}\in\mathscr{L}_{p,\alpha}$ there exists the decomposition

$$\hat{u}=c_{-1}\hat{u}_{-1}+c_0\hat{u}_0+\sum_{n=1}^{\infty}c_n^{+}\hat{u}_n^{+}+\sum_{n=1}^{\infty}c_n^{-}\hat{u}_n^{-}, \tag{3.3}$$

and this decomposition is unique. This decomposition is equivalent to the next two decompositions

$$u\left(t\right)=c_{-1}+\sum_{n=1}^{\infty}c_n^{+}\left(in\right)e^{int}+\sum_{n=1}^{\infty}c_n^{-}\left(-in\right)e^{-int}, \tag{3.4}$$

$$\lambda=-\pi c_{-1}+c_0+\sum_{n=1}^{\infty}c_n^{+}e^{-in\pi}+\sum_{n=1}^{\infty}c_n^{-}e^{in\pi}. \tag{3.5}$$

Following [3] we obtain that there exists the decomposition (3.4) and it is unique. Let us note that the decomposition (3.4) belongs to the space $ML^{p,\alpha}$ and since $L^{p,\alpha}\subset L_p$, then Hausdorff-Young inequality holds for the system $\left\{e^{int}\right\}_{n\in Z}$ in Morrey space $L^{p,\alpha}$. I.e., if $1<p\leq 2$ then

$$\left(|c_1|^q+\sum_{n=1}^{\infty}\left|c_n^{-}n\right|^q+\sum_{n=1}^{\infty}\left|c_n^{+}n\right|^q\right)^{1/q}\leq M\left\|u\right\|_{L_p},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Applying Hölder's inequality, we obtain

$$|c_{-1}|+\sum_{n=1}^{\infty}\left|c_n^{-}\right|+\sum_{n=1}^{\infty}\left|c_n^{+}\right|=|c_{-1}|+\sum_{n=1}^{\infty}\frac{1}{n}\left|nc_n^{-}\right|+\sum_{n=1}^{\infty}\frac{1}{n}\left|nc_n^{+}\right|\leq|c_{-1}|+$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\sum_{n=1}^{\infty}\left|nc_n^{-}\right|^q+\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\sum_{n=1}^{\infty}\left|nc_n^{+}\right|^q<+\infty.$$

In the case of $p > 2$ if $L^{p,\alpha} \subset L_p \subset L_2$ then

$$\left(|c_{-1}|^2 + \sum_{n=1}^{\infty} \left| c_n^- n \right|^2 + \sum_{n=1}^{\infty} \left| c_n^+ n \right|^2 \right)^{1/2} \leq M \left\| u \right\|_{L_2}$$

and similarly

$$|c_{-1}| + \sum_{n=1}^{\infty} \left| c_n^- \right| + \sum_{n=1}^{\infty} \left| c_n^+ \right| \leq |c_0| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^2 \left(\sum_{n=1}^{\infty} \left| n c_n^- \right|^2 + \sum_{n=1}^{\infty} \left| n c_n^+ \right|^2 \right) < +\infty.$$

So, we show that the series $\sum_{n=1}^{\infty} |c_n^{\pm}|$ is absolutely convergent. Therefore in the decomposition (3.5) the coefficient c_0 is uniquely defined. Thus, we have shown the existence and uniqueness of the decomposition (3.3) for $\forall \hat{u} \in \mathscr{L}_{p,\alpha}$. I.e. a system $\{\hat{u}_{-1}\} \bigcup \{\hat{u}_0\} \bigcup \{\hat{u}_n^{\pm}\}_{n \geq 1}$ forms a basis for $\mathscr{L}_{p,\alpha}$. We can easily calculate that for the operator

$$A\hat{u} = \lambda + \int_{-\pi}^{t} u(\tau)\, d\tau,$$

the following relations are true

$$A(\hat{u}_{-1}) = t, A(\hat{u}_0) = 1$$

$$A\left(\hat{u}_n^-\right) = e^{-int}, A\left(\hat{u}_n^+\right) = e^{int}.$$

If A is an isomorphism, then a system $t \bigcup \left\{ e^{int} \right\}_{n \in Z}$ forms a basis for $MW_{p,\alpha}^1$. The theorem is proved.

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References

1. Bilalov, B.T. – *The basis property of some systems of exponentials of cosines and sines* (Russian), translated from Differentsialnye Uravneniya, 26 (1990), no. 1, 10-16, 180 Differential Equations **26** (1990), no. 1, 8-13.
2. Bilalov, B.T. – *Basis properties of some systems of exponentials, cosines, and sines* (Russian), translated from Sibirsk. Mat. Zh., **45** (2004), no. 2, 264-273, Siberian Math. J. 45 (2004), no. 2, 214-221.
3. Bilalov, B.T.; Quliyeva, A.A. – *On basicity of exponential systems in Morrey-type spaces*, Internat. Journal of Math., **25** (2014), no. 6, 1450054, 10 pp, DOI:10.1142/S0129167X14500542.
4. Bilalov B.T.; Gasymov T.B.; Quliyeva A.A. – *On the solvability of the Riemann boundary value problem in Morrey-Hardy classes*, Turkish Journal of Mathematics. DOI: 10.3906/mat-1507-10 (accepted).
5. Chen, Y. – *Regularity of the solution to the Dirichlet problem in Morrey space*, J. Partial Differential Equations, **15** (2002), no. 2, 37-46.

6. Duoandikoetxea, J. – *Weights for maximal functions and singular integrals*, NCTS Summer School on Harmonic Analysis in Taiwan, 2005.
7. Giga, Y.; Miyakawa, T. – *Navier–Stokes flow in R^3 with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations, **14** (1989), no. 5, 577-618.
8. Israfilov, D.M.; Tozman, N.P. – *Approximation by polynomials in Morrey–Smirnov classes*, East J. Approx., **14** (2008), no. 3, 255-269.
9. Israfilov, D.M.; Tozman, N.P. – *Approximation in Morrey–Smirnov classes*, Azerbaijan J. Math., **1** (2011), no. 1, 99-113.
10. Kokilashvili, V.; Meskhi, A. – *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Armen. J. Math., **1** (2008), no. 1, 18-28.
11. Ky, N.X. – *On approximation by trigonometric polynomials in $L_{p,u}$-spaces*, Studia Sci. Math. Hungar, **28** (1993), no. 1-2, 183-188.
12. Lemarié-Rieusset, P.G. – *Some remarks on the Navier–Stokes equations in R^3*, J. Math. Phys., **39** (1998), no. 8, 4108-4118.
13. Mazzucato, A.L. – *Decomposition of Besov-Morrey spaces*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 279-294, Contemp. Math., **320**, Amer. Math. Soc., Providence, RI, 2003.
14. Moiseev, E.I. – *The basis property for systems of sines and cosines* (Russian), Dokl. Akad. Nauk SSSR, **275** (1984), no. 4, 794-798.
15. Peetre, J. – *On the theory of $L_{p,\lambda}$ spaces*, J. Functional Analysis, **4** (1969), 71-87.
16. Samko, N. – *Weighted Hardy and singular operators in Morrey spaces*, Journal of Mathematical Analysis and Applications **350** (2009), no. 1, 56-72.
17. Zorko, C.T. – *Morrey space*, Proc. Amer. Math. Soc. **98** (1986), no. 4, 586–592.

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