Turk J Math
35 (2011) , 47 – 54.

doi:10.3906/mat-0908-160

On the stability of basisness in L_p $(1 < p < +\infty)$ of cosines and sines

Ali A. Huseynli

Abstract

We study the basis properties in $L_p(0,\pi)$ $(1 < p < \infty)$ of the solution system of Sturm–Liouville equations with different types of initial conditions. We first establish some results on the stability of the basis property of cosines and sines in $L_p(0,\pi)$ $(1 < p < \infty)$ and then show that the solution system above forms a basis in $L_p(0,\pi)$ if and only if certain cosine system (or sine system, depending on type of initial conditions) forms a basis in $L_p(0,\pi)$.

Key Words: Bases of cosines and sines, Sturm–Liouville equation

Denote by $u(x,\lambda)$ and $v(x,\lambda)$ the solutions of Sturm–Liouville equation

$$-y'' + q(x)\,y = \lambda^2 y$$

satisfying the initial conditions

$$y(a) = 1, \quad y'(a) = \sigma$$

and

$$y(a) = 0, \quad y'(a) = \lambda,$$

respectively.

The problem of finding complex sequences $\{\lambda_n\}$ for which the systems $\{u(x,\lambda_n)\}$ and $\{v(x,\lambda_n)\}$ form a basis in some functional space is very important. In [1] it was proved that the system $\{u(x,\lambda_n)\}$ (respectively, $\{v(x,\lambda_n)\}$) forms a Riesz basis in $L_2(0,\pi)$ if and only if the system $\{\cos\lambda_n x\}$ (respectively, $\{\sin\lambda_n x\}$) forms a Riesz basis in $L_2(0,\pi)$. In this paper we present a generalization of this result for $L_p(0,\pi)$ $(1 \le p < +\infty)$ spaces. More precisely, we prove that the system $\{u(x,\lambda_n)\}$ (respectively, $\{v(x,\lambda_n)\}$) forms a basis in $L_p(0,\pi)$ $(1 \le p < +\infty)$ if and only if the system $\{\cos\lambda_n x\}$ (respectively, $\{\sin\lambda_n x\}$) forms a basis in $L_p(0,\pi)$. We also present an elementary proof based on transformation operators from the spectral theory of differential operators (see, e.g., [6]).

The structure (e.g. completeness, basis or frame properties) of the systems $\{\cos\lambda_n x\}$ or $\{\sin\lambda_n x\}$ in $L_p(0,\pi)$ is closely related with the structure of exponential systems $\{e^{\pm i\lambda_n x}\}$ in $L_p(-\pi,\pi)$. The study of

2000 *AMS Mathematics Subject Classification:* 34L10, 42C15.

exponential systems, often referred to as the theory of nonharmonic Fourier series (see [4, 7, 9, 10, 11]), has its origins in the classical works of R. Paley and N. Wiener [7] and N. Levinson [4]. One of the famous early results in the theory is that the basis property of the trigonometric system $\left\{e^{inx}\right\}_{-\infty}^{+\infty}$ is stable in $L_2(-\pi,\pi)$ in the sense that the system $\left\{e^{i\lambda_n x}\right\}_{-\infty}^{+\infty}$ will always form a Riesz basis for $L_2(-\pi,\pi)$ if $|\lambda_n - n| \le L < 1/4$. M.I. Kadec [2], and R. M. Redheffer and R. M. Young [8] have shown $1/4$ to be optimal.

The theory for sequences of cosines and sines appears to be less complete. Therefore, we first investigate such sequences in Section 2. We prove a theorem on the stability of the basis property of cosines and sines in $L_p(0,\pi)$ $(1 < p < +\infty)$, which is a generalization of the corresponding theorem in [1], where only $L_2(0,\pi)$ case was considered. At the same time we present an elementary proof.

1. Necessary notations, definitions and facts

By $\|\cdot\|_p$ we denote the norm in the space L_p. Let $\mathcal{E} = \{e_n\}_{n=1}^{\infty}$ be a basis in the space L_p. We denote by $\mathcal{K}_p(\mathcal{E})$ the set of coefficients of the basis $\mathcal{E}$, i.e., the set of all sequences $\{c_n\}_{n=1}^{\infty}$ of complex numbers, for which the series $\sum\limits_{n=1}^{\infty} c_n e_n$ is convergent in L_p. It is well known that, if we define linear operations coordinate-wise in $\mathcal{K}_p(\mathcal{E})$ and for $\{c_n\}_{n=1}^{\infty} \in \mathcal{K}_p(\mathcal{E})$ we take by definition $\|\{c_n\}_{n=1}^{\infty}\| \overset{def}{=} \sup\limits_N \left\| \sum\limits_{n=1}^{N} c_n e_n \right\|_p$, then $\mathcal{K}_p(\mathcal{E})$ becomes a Banach space (see, e.g., [5]).

Definition 1 *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The sequence $\{\lambda_n\}_{n=1}^{\infty}$ is called separated if there exists $\varepsilon > 0$ such that $\inf\limits_{\substack{n,k\in N \\ n\neq k}} |\lambda_n - \lambda_k| \ge \varepsilon$.*

Definition 2 *A system $\{f_n(x)\}_{n=1}^{\infty}$, $f_n \in L_p(a,b)$ is called q-Hilbert system in the space $L_p(a,b)$ if there exists $m > 0$, such that for every finite system $\{c_n\}$ of complex numbers*

$$\left(\sum_n |c_n|^q\right)^{1/q} \le m \cdot \left\| \sum_n c_n f_n \right\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

It follows from the theorem of Riesz that, in case $1 < p \le 2$ every uniformly bounded and orthonormal system of functions in $L_p(a,b)$ is q–Hilbert system in the space $L_p(a,b)$ [12].

Lemma 1 *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If the system $\{\cos\lambda_n x\}_{n=1}^{\infty}$ is q-Hilbert system in the space $L_p(0,\pi)$, $1 < p < \infty$, then the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is separated.*

Proof. Since $|\cos\lambda_n x - \cos\lambda_k x| \le \pi \cdot |\lambda_n - \lambda_k|$ and the system $\{\cos\lambda_n x\}_{n=1}^{\infty}$ is q–Hilbert system, we have

(for $n \neq k$)

$$2^{1/q} \leq m \cdot \left(\int_0^\pi |\cos \lambda_n x - \cos \lambda_k x|^p \, dx \right)^{1/p} \leq$$

$$\leq m \cdot \left(\int_0^\pi \pi^p \, |\lambda_n - \lambda_k|^p \, dx \right)^{1/p} = m \cdot \pi^{1+1/p} \, |\lambda_n - \lambda_k|$$

which demonstrates that $\{\lambda_n\}$ is separated. □

For the proof of our main theorem (Theorem 4) we will need the following results.

Lemma 2 *Let $\{e_i\}_{i=1}^{\infty}$ be a basis of the Banach space B. If an arbitrary finite number of elements are replaced by other elements of the space B, then the new system is either basis of B, or is neither complete, nor minimal in B.*

Theorem 1 ([3], [4], [10]) *If the system $\left\{e^{i\lambda_k x}\right\}$ is complete in $L_p(-a,a)$ or in $C[-a,a]$, and if an arbitrary number n of functions are removed from this system and replaced by n other functions $e^{i\mu_j x}$ $(j = 1, 2, ..., n)$ where $\mu_1, \mu_2, ..., \mu_n$ are arbitrary different complex numbers not equal to any λ_k, then the new system will be complete in the same sense as the original system.*

Theorem 2 ([10]) *Let $\{\lambda_n\}_{n=1}^{\infty}$ be an arbitrary sequence of complex numbers, such that $\lambda_n \neq 0$, $\lambda_n \neq \lambda_m$ for $n \neq m$ and $-\lambda_m \notin \{\lambda_n\}_{n=1}^{\infty}$ for all m. The system $1 \cup \{\cos \lambda_n t\}_{n=1}^{\infty}$ (respectively $\{\cos \lambda_n t\}_{n=1}^{\infty}$) is complete in $L_p(0,a)$ $(1 \leq p < +\infty)$ if and only if the system $e^{\pm i\mu t} \cup \left\{e^{\pm i\lambda_n t}\right\}_{n=1}^{\infty}$, $\mu \neq 0$, $\pm\mu \notin \{\lambda_n\}_{n=1}^{\infty}$ (respectively, $\left\{e^{\pm i\lambda_n t}\right\}_{n=1}^{\infty}$) is complete in $L_p(-a,a)$.*

Theorem 3 ([10]) *Let $\{\lambda_n\}_{n=1}^{\infty}$ be an arbitrary sequence of complex numbers, such that $\lambda_n \neq 0$, $\lambda_n \neq \lambda_m$ for $n \neq m$ and $-\lambda_m \notin \{\lambda_n\}_{n=1}^{\infty}$ for all m. The system $\{\sin \lambda_n t\}_{n=1}^{\infty}$ is complete in $L_p(0,a)$ $(1 \leq p < +\infty)$ if and only if the system $1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n=1}^{\infty}$ is complete in $L_p(-a,a)$.*

Theorems 1 and 2 imply the following result.

Corollary 1 *If the system $\{\cos \lambda_k x\}$ is complete in $L_p(0,\pi)$ or in $C[0,\pi]$, and if an arbitrary number n of functions are removed from this system and replaced by n other functions $\cos \mu_j x$ $(j = 1, 2, ..., n)$, where $\mu_1, \mu_2, ..., \mu_n$ are arbitrary complex numbers such that $\mu_i \neq \pm\mu_j$ for $i \neq j$, $i, j = 1, 2, ...n$ and μ_i are not equal to any $\pm\lambda_k$, then the new system will be complete in the same sense as the original system.*

Theorems 1 and 3 imply that Corollary 1 is also true for the system $\{\sin \lambda_k x\}$.

2. Stability of basisness of cosines and sines

Theorem 4 *Let $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers with $\lambda_i \neq \lambda_j$, $\mu_i \neq \mu_j$, for $i \neq j$ and assume that, for some $1 < p < \infty$ the inequality*

$$\sum_{n=0}^{\infty} |\lambda_n - \mu_n|^{\alpha} < \infty$$

holds, where $\alpha = \min(p, q)$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\{\cos \lambda_n x\}_{n=0}^{\infty}$ is a basis in the space $L_p(0, \pi)$ isomorphic to the basis $\{\cos nx\}_{n=0}^{\infty}$, then the system $\{\cos \mu_n x\}_{n=0}^{\infty}$ is also a basis in $L_p(0, \pi)$, isomorphic to the basis $\{\cos \lambda_n x\}_{n=0}^{\infty}$.

Proof. First consider the case $1 < p \leq 2$. Then $q \geq 2$ and $\alpha = p$. Denote $\varphi_n(x) = \cos \lambda_n x$, $\psi_n(x) = \cos \mu_n x$, $n = 0, 1, 2, \ldots$.

Since

$$|\varphi_n(x) - \psi_n(x)| = |\cos \lambda_n x - \cos \mu_n x| \leq \pi \cdot |\lambda_n - \mu_n| \tag{1}$$

then

$$\|\varphi_n - \psi_n\|_p^p \leq \int_0^{\pi} \pi^p |\lambda_n - \mu_n|^p \, dx = \pi^{p+1} \cdot |\lambda_n - \mu_n|^p .$$

Due to the condition of the theorem, the series $\sum_{n=0}^{\infty} |\lambda_n - \mu_n|^p$ is convergent, hence the series $\sum_{n=0}^{\infty} \|\varphi_n - \psi_n\|_p^p$ is also convergent.

Since the system $\{\varphi_n\}_{n=0}^{\infty}$ is a basis, isomorphic to the basis $\{\cos nx\}_{n=0}^{\infty}$ in the space $L_p(0, \pi)$, then the set $\mathcal{K}_p(\{\varphi_n\}_{n=0}^{\infty})$ coincides with the set $\mathcal{K}_p(\{\cos nx\}_{n=0}^{\infty})$:

$$\mathcal{K}_p(\{\varphi_n\}_{n=0}^{\infty}) \equiv \mathcal{K}_p(\{\cos nx\}_{n=0}^{\infty}) \stackrel{def}{=} \mathcal{K}_p.$$

According to the Hausdorf-Young theorem (see, e.g. [12]) we have

$$\exists M_p > 0, \ \forall c = (c_0, c_1, ..., c_n, ...) \in \mathcal{K}_p :$$

$$\left(\sum_{n=0}^{\infty} |c_n|^q \right)^{1/q} \leq M_p \cdot \left\| \sum_{n=0}^{\infty} c_n \cos nx \right\|_p . \tag{2}$$

Since the bases $\{\varphi_n\}_{n=0}^{\infty}$ and $\{\cos nx\}_{n=0}^{\infty}$ are isomorphic, then

$$\exists K > 0, \ \forall c = (c_0, c_1, ..., c_n, ...) \in \mathcal{K}_p :$$

$$\left\| \sum_{n=0}^{\infty} c_n \cos nx \right\|_p \leq K \cdot \left\| \sum_{n=0}^{\infty} c_n \varphi_n \right\|_p . \tag{3}$$

We fix a natural number m satisfying the condition

$$\sum_{n=m}^{\infty} \|\varphi_n - \psi_n\|_p^p < (2M_pK)^{-p}. \tag{4}$$

Consider the system $\{f_n\}_{n=0}^{\infty} \subset L_p(0,\pi)$:

$$f_n = \begin{cases} \varphi_n, & n = 0, 1, ..., m-1, \\ \psi_n, & n = m, m+1, ... \end{cases}$$

Inequalities (2), (3) and (4) imply that for any finite sequence $(c_0, c_1, ..., c_k)$, $k \geq m$

$$\left\| \sum_{n=0}^{k} c_n (f_n - \varphi_n) \right\|_p \leq \sum_{n=0}^{k} |c_n| \cdot \|f_n - \varphi_n\|_p \leq$$
$$\leq \left(\sum_{n=0}^{k} |c_n|^q \right)^{1/q} \cdot \left(\sum_{n=0}^{k} \|f_n - \varphi_n\|_p^p \right)^{1/p} \leq$$
$$\leq M_p \cdot K \cdot \left(\sum_{n=m}^{k} \|\psi_n - \varphi_n\|_p^p \right)^{1/p} \cdot \left\| \sum_{n=0}^{k} c_n \varphi_n \right\|_p \leq \frac{1}{2} \cdot \left\| \sum_{n=0}^{k} c_n \varphi_n \right\|_p.$$

For $k < m$ the truth of this inequality is obvious, since in this case $\sum_{n=0}^{k} c_n (f_n - \varphi_n) = 0$. According to Paley-Wiener theorem [11] the system $\{f_n\}_{n=0}^{\infty}$ forms a basis in the space $L_p(0,\pi)$, isomorphic to the basis $\{\varphi_n\}_{n=0}^{\infty}$.

Now, replacing the functions $f_0, f_1, ..., f_{m-1}$ by the functions $\psi_0, \psi_1, ..., \psi_{m-1}$ and taking into account that $\mu_i \neq \mu_j$ for $i \neq j$, from Corollary 1 and Lemma 2 we obtain that the system $\{\psi_n\}_{n=0}^{\infty}$ is a basis in the space $L_p(0,\pi)$, isomorphic to the basis $\{\varphi_n\}_{n=0}^{\infty}$.

Now, consider the case $p > 2$. In this case $q < 2$ and $\alpha = q$. Then it is known that $L_p \subset L_q$ and there exists a constant C_p, such that for all $x \in L_p$

$$\|x\|_q \leq C_p \cdot \|x\|_p. \tag{5}$$

We fix a natural number m, satisfying the inequality

$$\sum_{n=m}^{\infty} \|\varphi_n - \psi_n\|_p^q < (2M_q \cdot K \cdot C_p)^{-q} \tag{4*}$$

(the inequality (1) and the condition of the theorem imply that in this case the series $\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^q$ converges). As we did above, consider the system $\{f_n\}_{n=0}^{\infty} \subset L_p(0,\pi)$:

$$f_n = \begin{cases} \varphi_n, & n = 0, 1, ..., m-1, \\ \psi_n, & n = m, m+1, ... \end{cases}$$

From $(2),(3),(5)$ and (4^*) we have

$$\left\|\sum_{n=0}^{k} c_n (f_n - \varphi_n)\right\|_p \le \sum_{n=0}^{k} |c_n| \cdot \|f_n - \varphi_n\|_p \le \left(\sum_{n=0}^{k} |c_n|^p\right)^{1/p}$$
$$\times \left(\sum_{n=0}^{k} \|f_n - \varphi_n\|_p^q\right)^{1/q} \le M_q \cdot \left(\sum_{n=m}^{k} \|\psi_n - \varphi_n\|_p^q\right)^{1/q} \cdot \left\|\sum_{n=0}^{k} c_n \cos nx\right\|_q$$
$$\le M_q \cdot C_p \cdot \left(\sum_{n=m}^{k} \|\psi_n - \varphi_n\|_p^q\right)^{1/q} \times \left\|\sum_{n=0}^{k} c_n \cos nx\right\|_p$$
$$\le M_q \cdot C_p \cdot K \cdot \left(\sum_{n=m}^{k} \|\psi_n - \varphi_n\|_p^q\right)^{1/q} \cdot \left\|\sum_{n=0}^{k} c_n \varphi_n\right\|_p$$
$$\le M_q \cdot K \cdot C_p \cdot \frac{1}{2 \cdot M_q \cdot K \cdot C_p} \cdot \left\|\sum_{n=0}^{k} c_n \varphi_n\right\|_p = \frac{1}{2} \cdot \left\|\sum_{n=0}^{k} c_n \varphi_n\right\|_p .$$

For $k < m$ the truth of this inequality is obvious. Now applying the same arguments, that we have done for the case $p \le 2$, we obtain that, the system $\{\psi_n\}_{n=0}^{\infty}$ is a basis in $L_p(0,\pi)$, isomorphic to the basis $\{\varphi_n\}_{n=0}^{\infty}$. This completes the proof. □

In particular, for $p = 2$ we obtain that, if the system $\{\cos \lambda_n x\}_{n=0}^{\infty}$ is a Riesz basis in $L_2(0,\pi)$ and the condition $\sum_{n=1}^{\infty} |\lambda_n - \mu_n|^2 < \infty$ holds, then the system $\{\cos \mu_n x\}_{n=0}^{\infty}$ also forms a Riesz basis in $L_2(0,\pi)$. This result was obtained in [1] by other methods.

Lemma 1 and Theorem 4 are true with $\{\sin \lambda_n x\}$ in place of $\{\cos \lambda_n x\}$ if, in Theorem 4 we replace "nonnegative" by "positive". We omit the details.

3. Stability of bases of solutions to Sturm–Liouville equations

3.1. The case of initial conditions $y(0) = 1$, $y'(0) = \sigma$

We consider the following Cauchy problem:

$$-y'' + q(x)\, y = \lambda^2 y, \quad 0 \le x \le \pi, \tag{6}$$

$$y(0) = 1,\ y'(0) = \sigma, \tag{7}$$

where $q(x)$ is an integrable function on $[0,\pi]$ and σ is a constant. We denote by $y(x,\lambda)$ the solution of the problem $(6) - (7)$. We are interested in the question: for which sequences $\{\lambda_n\}_{n=1}^{\infty}$ the system of functions $\{y(x,\lambda_n)\}_{n=1}^{\infty}$ forms a basis in $L_p(0,\pi)$, $1 < p < \infty$? The answer to this question is given by the following theorem.

Theorem 5 *The system of functions $\{y(x,\lambda_n)\}_{n=1}^{\infty}$ forms a basis in the space $L_p(0,\pi)$ if and only if the system $\{\cos \lambda_n x\}_{n=1}^{\infty}$ forms a basis in the space $L_p(0,\pi)$.*

Proof. It is well known that the following representations are true:

$$y(x,\lambda)=\cos\lambda x+\int_0^x K(x,t)\cos\lambda t\;dt, \tag{8}$$

$$\cos\lambda x=y(x,\lambda)+\int_0^x L(x,t)\,y(t,\lambda)\,dt, \tag{9}$$

where $K(x,t)$ and $L(x,t)$ are continuous functions (see, e.g. [6]). If we denote by $I+K$ and $I+L$ the operators defined by the right hand sides of the equality (8) and (9) respectively, then it is clear that, the operator $I+K$ is continuously invertible and $(I+K)^{-1}=I+L$. Now the validity of the theorem follows from the equality $y(x,\lambda)=(I+K)\cos\lambda x$. □

In particular, when $p=2$ we have that the system $\{y(x,\lambda_n)\}_{n=1}^{\infty}$ forms a Riesz basis in $L_2(0,\pi)$ if and only if the system $\{\cos\lambda_n x\}_{n=1}^{\infty}$ forms a Riesz basis in $L_2(0,\pi)$. This result was obtained in [1] by other methods.

3.2. The case of initial conditions $y(0)=1$, $y'(0)=\lambda$

Let $y(x)=y(x,\lambda)$ be the solution of the Sturm–Liouville equation (6) with the initial conditions

$$y(0)=0,\quad y'(0)=\lambda.$$

where $q(x)$ is an integrable function on $[0,\pi]$.

Theorem 6 *The system of functions* $\{y(x,\lambda_n)\}_{n=1}^{\infty}$ *forms a basis in the space* $L_p(0,\pi)$ *if and only if the system* $\{\sin\lambda_n x\}_{n=1}^{\infty}$ *forms a basis in the space* $L_p(0,\pi)$.

Proof. The following representations are true:

$$y(x,\lambda)=\sin\lambda x+\int_0^x K(x,t)\sin\lambda t dt, \tag{10}$$

$$\sin\lambda x=y(x,\lambda)+\int_0^x L(x,t)\,y(t,\lambda)\,dt, \tag{11}$$

where $K(x,t)$ and $L(x,t)$ are continuous functions (see, e.g. [6]). If we denote by $I+K$ and $I+L$ the operators defined by the right hand sides of the equality (10) and (11) respectively, then it is clear that, the operator $I+K$ is continuously invertible and $(I+K)^{-1}=I+L$. Now the validity of the theorem follows from the equality $y(x,\lambda)=(I+K)\sin\lambda x$. □

In particular, when $p=2$ we have that the system $\{y(x,\lambda_n)\}_{n=1}^{\infty}$ forms a Riesz basis in $L_2(0,\pi)$ if and only if the system $\{\sin\lambda_n x\}_{n=1}^{\infty}$ forms a Riesz basis in $L_2(0,\pi)$. This result was obtained in [1].

References

[1] He, X., Volkmer, H.: Riesz bases of solutions of Sturm–Lioville equations. The Journal of Fourier Analysis and Applications 7(3), 297-307 (2001).

[2] Kadec, M.I.: The exact value of the Paley-Wiener constant. Dokl. Akad. Nauk SSSR 155, 1253-1254 (1964), English translation: Sov. Math. Dokl. 5, 559-561 (1964).

[3] Levin, B.Ya.: Distribution of zeros of entire functions. Amer. Math. Soc. Translations of Mathemaical Monographs, Vol.5 (1980).

[4] Levinson, N.: Gap and Density Theorems. AMS. Col. Public. 26 (1940).

[5] Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces, Vol. I. Springer 1977.

[6] Marchenko, V.A.: Sturm–Liouville operators and their applications. Kiev. Naukova Dumka 1977.

[7] Paley, R.E., Wiener, N.: Fourier transforms in the complex domain. AMS. Col. Public. Vol. 19. 1934.

[8] Redheffer, R.M., Young, R.M.: Completeness and basis properties of complex exponentials. Trans. Amer. Math. Soc. 277, 93-111 (1983).

[9] Sedletskii, A.M.: Fourier transforms and approximations. Amsterdam. Gordon and Breach Science Publ. 2000.

[10] Sedletskii, A.M.: Classes of analytic Fourier transforms and exponential approximations. Moscow. Physmatlit 2005.

[11] Young, R.M.: An Introduction to Nonharmonic Fourier Series. Academic Press 1980.

[12] Zygmund, A.: Trigonometric series, Vol. II. Cambridge University Press 1959.

Ali A. HUSEYNLI
Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan,
9 F.Agayev str., AZ1141, Baku-AZERBAIJAN
e-mail: aahuseynli@gmail.com

Received 06.08.2009