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Spectral functions for classical and generalized Fourier transforms

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The talk will consist of two parts. The first part deals with spectral and pseudospectral functions for generalized Fourier transform corresponding to symmetric differential system

$$Jy' - A(t)y = \lambda H(t)y. \tag{1}$$

It is assumed that $n \times n$-matrix coefficients $J(= -J^* = -J^{-1})$ and $A(t) = A^*(t),\ H(t) \geq 0$ in (1) are defined on an interval $\mathcal{I} = [a, b),\ -\infty < a < b \leq \infty$, and integrable on each compact subinterval $[a, \beta] \subset \mathcal{I}$. Denote by $L^2_H(\mathcal{I}, \mathbb{C}^n)$ the Hilbert space of vector-functions $f : \mathcal{I} \to \mathbb{C}^n$ satisfying $\int\limits_{\mathcal{I}} (H(t)f(t), f(t))\, dt < \infty$ and by $N_\pm$ deficiency indices of the system (1), i.e., the number of its linearly independent solutions $y \in L^2_H(\mathcal{I}, \mathbb{C}^n)$ for $\lambda \in \mathbb{C}_\pm$.

Let $m \leq n$ and let $\varphi(t, \lambda)(\in \mathbb{C}^{n\times m})$ be a matrix solution of (1) with $\varphi(0, \lambda) = \text{const}$. Then the generalized Fourier transform of a vector-function $f(\cdot) \in L^2_H(\mathcal{I}, \mathbb{C}^n)$ is a vector-function $\widehat{f}(\cdot) : \mathbb{R} \to \mathbb{C}^m$ given by

$$\widehat{f}(s) = \int\limits_{\mathcal{I}} \varphi^*(t, s)H(t)f(t)\, dt. \tag{2}$$

We define a spectral (resp. pseudospectral) function of the system with respect to the transform (2) as a matrix-valued distribution function $\sigma(s),\ s \in \mathbb{R}$, of the dimension $n_\sigma := m$ such that the operator $V_\sigma : L^2_H(\mathcal{I}, \mathbb{C}^n) \to L^2(\sigma; \mathbb{C}^m)$ defined by $(V_\sigma f)(s) := \widehat{f}(s),\ f \in L^2_H(\mathcal{I}, \mathbb{C}^n)$, is an isometry (resp. a partial isometry with the minimally possible kernel). Moreover, we find the minimally possible dimension of a spectral function and parameterize all spectral and pseudospectral functions of every possible dimension n_σ. In the case $N_+ = N_-$ such a parametrization is given by the Redheffer transform

$$m_\tau(\lambda) = m_0(\lambda) + S(\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}(\lambda))^{-1}C_1(\lambda)S^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \tag{3}$$

and by the Stieltjes inversion formula

$$\sigma_\tau(s) = \lim_{\delta\to+0} \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_\tau(u + i\varepsilon)\, du. \tag{4}$$

for the Nevanlinna matrix-function $m_\tau(\lambda)$ (the m-function of the system). Here $m_0(\lambda),\ S(\lambda)$ and $\dot{M}(\lambda)$ are matrix-valued coefficients defined in terms of respective matrix solutions of the system and $\tau = \{C_0(\lambda), C_1(\lambda)\},\ \lambda \in \mathbb{C}_+$, is a Nevanlinna pair (a boundary parameter) satisfying the following admissibility conditions:

$$\begin{aligned}&\lim_{y\to\infty} \tfrac{1}{iy}(C_0(iy) - C_1(iy)\dot{M}(iy))^{-1}C_1(iy) = 0,\\&\lim_{y\to\infty} \tfrac{1}{iy}\dot{M}(iy)(C_0(iy) - C_1(iy)\dot{M}(iy))^{-1}C_0(iy) = 0.\end{aligned} \tag{5}$$

With a certain modification the parametrization (3), (4) holds in the case $N_+ \neq N_-$ as well.

Assume now that $N_- \leq N_+ = n$ (this means that N_+ is maximally possible). For this case we define the monodromy matrix $B(\lambda)$ as a singular boundary value of the matrizant $Y(t,\lambda)$ at the endpoint b and parameterize all spectral and pseudospectral functions $\sigma(\cdot)$ of any possible dimension n_σ by means of the linear-fractional transform

$$m_\tau(\lambda) = (C_0(\lambda)w_{11}(\lambda) + C_1(\lambda)w_{21}(\lambda))^{-1}(C_0(\lambda)w_{12}(\lambda) + C_1(\lambda)w_{22}(\lambda))$$

and formula (4). Here $w_{ij}(\lambda)$ are the matrix coefficients defined in terms of $B(\lambda)$ and $\tau = \{C_0(\lambda), C_1(\lambda)\}$ is the same as in (3); moreover, the admissibility conditions (5) can be written as

$$\begin{aligned}&\lim_{y\to+\infty} \tfrac{1}{iy}w_1(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_1(iy) = 0\\&\lim_{y\to+\infty} \tfrac{1}{iy}w_3(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_0(iy) = 0\end{aligned}$$

It turns out that the matrix $W(\lambda) = (w_{ij}(\lambda))_{i,j=1}^2$ has the properties similar to those of the resolvent matrix in the extension theory of symmetric operators.

The specified results develop the results by Arov and Dym; A. Sakhnovich, L. Sakhnovich and Roitberg; Langer and Textorius.

The second part of the talk is devoted to the classical vector-valued Fourier transform

$$\widehat{f}(s) = \int_{\mathbb{R}} e^{its} f(t)\, dt. \tag{6}$$

of the vector-valued function $f(t)$. Assume that $\widetilde{\mathcal{I}} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n\}$ is a system of intervals $\mathcal{I}_j = \langle a_j, b_j\rangle,\ -\infty \leq a_j < b_j \leq \infty$. Denote by $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{I}})$ the set of all vector-functions

$$f(t) = \{f_1(t), f_2(t), \dots, f_n(t)\} (\in \mathbb{C}^n), \quad t \in \mathbb{R},$$

such that $\int\limits_{\mathbb{R}} ||f(t)||^2\, dt < \infty$ and support of a coordinate function f_j lies in $\mathcal{I}_j$. For each function $f \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{I}})$ with compact support equality (6) defines the vector-valued Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}^n$ of f. A matrix-valued distribution function $\sigma : \mathbb{R} \to \mathbb{C}^{n\times n}$ will be called a spectral function for the vector-valued Fourier transform (6) (with respect to $\widetilde{\mathcal{I}}$) if the following Parseval equality holds:

$$\int_{\mathbb{R}} (d\sigma(s)\widehat{f}(s), \widehat{f}(s)) = \int_{\mathbb{R}} ||f(t)||^2\, dt, \quad f \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{I}}).$$

The set of all such spectral functions we denote by $SF_n(\widetilde{\mathcal{I}}) = SF_n(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$. If $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{I}})$, then for each $f \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{I}})$ the inverse Fourier transform is

$$f(t) = \chi_{\widetilde{\mathcal{I}}}(t) \int\limits_{\mathbb{R}} e^{-its}\, d\sigma(s) \widehat{f}(s),$$

where $\chi_{\widetilde{\mathcal{I}}}(t) = \mathrm{diag}(\chi_{\mathcal{I}_1}(t), \chi_{\mathcal{I}_2}(t), \ldots, \chi_{\mathcal{I}_n}(t))$ ($\chi_{\mathcal{I}_j}$ is the indicator of $\mathcal{I}_j$).

In the particular case $n = 1$ system $\widetilde{\mathcal{I}}$ consists of a unique interval $\mathcal{I} = \langle a, b \rangle$ and equality (6) defines the classical $\mathbb{C}$-valued Fourier transform $\widehat{f}$ of a scalar function $f \in L^2(\mathbb{R})$ with support belonging to $\langle a, b \rangle$ (the set of such functions we denote by $L^2(\mathbb{R}; \langle a, b \rangle)$). The set $SF(\langle a, b \rangle)$ of spectral functions for this transform consists of scalar distribution functions $\sigma(\cdot)$ such that the Parseval equality

$$\int\limits_{\mathbb{R}} |\widehat{f}(s)|^2\, d\sigma(s) = \int\limits_{\mathbb{R}} |f(t)|^2\, dt, \quad f \in L^2(\mathbb{R}; \langle a, b \rangle) \tag{7}$$

holds; moreover, the inverse Fourier transform is

$$f(t) = \chi_{\mathcal{I}}(t) \int\limits_{\mathbb{R}} e^{-its} \widehat{f}(s)\, d\sigma(s). \tag{8}$$

A parametrization of the set $SF([0, b])$ in the case of a compact interval $[0, b]$ is given by the following theorem.

Theorem 1. Let $0 < b < \infty$. Then the equalities

$$m_\varphi(\lambda) = \frac{i}{2} \cdot \frac{e^{-i\lambda b} + \varphi(\lambda)}{e^{-i\lambda b} - \varphi(\lambda)}, \quad \lambda \in \mathbb{C}_+$$

$$\sigma_\varphi(s) = \lim_{\delta \to +0} \lim_{y \to +0} \frac{1}{\pi} \int\limits_{-\delta}^{s-\delta} \mathrm{Im}\, \mathrm{m}_\varphi(\mathrm{x} + \mathrm{iy})\, \mathrm{dx}$$

establish a bijective correspondence $\sigma(s) = \sigma_\varphi(s)$ between all holomorphic functions $\varphi(\lambda)$, $\lambda \in \mathbb{C}_+$, with $|\varphi(\lambda)| \leq 1$ and all scalar spectral functions $\sigma(\cdot) \in SF([0, b])$.

In the case $\varphi(\lambda) \equiv 1$ the spectral function $\sigma_\varphi(s)$ is a jump function and equalities (6) and (8) give an expansion of a function $f \in L^2(\mathbb{R}; [0, b])$ into the Fourier series on $[0, b]$. In the case $\varphi(\lambda) \equiv 0$ one has $\sigma_\varphi(s) = \frac{1}{2\pi} s$ and equality (8) turns into the classical inverse Fourier − Plancherel transform of a function $f \in L^2(\mathbb{R}; [0, b])$. Moreover, according to Theorem 1 there exist infinitely many spectral functions $\sigma(\cdot) \in SF[0, b]$. At the same time we show that in the case $\mathcal{I} = \mathbb{R}$ the set $SF(\mathbb{R})$ consists of the unique spectral function $\sigma(s) = \frac{1}{2\pi} s$ and equality (7) turns into the classical Parseval equality

$$\tfrac{1}{2\pi} \int\limits_{\mathbb{R}} |\widehat{f}(s)|^2\, ds = \int\limits_{\mathbb{R}} |f(t)|^2\, dt,$$

which holds according to the Plancherel theorem.

A parametrization of spectral functions $\sigma(\cdot) \in SF_2([0,\infty),(-\infty,0])$ is given by the following theorem.

Theorem 2. Let $\mathbf{C}_{\mathbb{R}}$ be the set of all complex-valued functions F on $\mathbb{R}$ admitting the representation

$$F(x) := \lim_{y \to +0} K(x+iy) \quad \text{(a.e. on } \mathbb{R}).$$

with a holomorphic function $K(\cdot)$ defined on an upper half-plane $\mathbb{C}_+$ and satisfying $|K(\lambda)| \leq 1,\ \lambda \in \mathbb{C}_+$. Then the equalities

$$\Sigma_F(x) = \frac{1}{2\pi}\begin{pmatrix} 1 & \overline{F(x)} \\ F(x) & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \quad \text{and} \quad \sigma_F(s) = \int_0^s \Sigma_F(x)\,dx \tag{9}$$

give a bijective correspondence $\sigma(\cdot) = \sigma_F(\cdot)$ between all functions $F \in \mathbf{C}_{\mathbb{R}}$ and all spectral functions $\sigma(\cdot) \in SF_2([0,\infty),(-\infty,0])$.

Theorem 2 shows that each spectral function $\sigma(\cdot) \in SF_2([0,\infty),(-\infty,0])$ is absolutely continuous with the matrix density $\Sigma_F(x)$ defined by the first equality in (9).

We parameterize also spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{I}})$ for other classes of $\widetilde{\mathcal{I}}$.

The results of the talk are partially specified in [1], [2].

References

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[2] V.I.Mogilevskii, *Pseudospectral functions of various dimensions for symmetric systems with the maximal deficiency index*, J. Math. Sciences **229** (2018), no. 1, 51–84.