Complex Analysis and Operator Theory
https://doi.org/10.1007/s11785-019-00936-9

Complex Analysis and Operator Theory

Properties of the Discrete Hilbert Transform

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Received: 11 December 2017 / Accepted: 10 June 2019
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Abstract

The asymptotic behavior of the distribution function of the Hilbert transform of sequences from the class l_1 is studied. The concept of Q-summability of series is introduced; using this notion, it is shown that the Hilbert transform of a sequence from the class l_1 is Q-summable and is Q-sum is zero.

Keywords Discrete Hilbert transform · Asymptotic behavior of the distribution function · Q-integral · Q-summability

Mathematics Subject Classification 44A15 · 26A39 · 40G99

1 Introduction

Let $\{b_n\}_{n\in Z} \in l_1$. The sequence

$$\tilde{b}_n = \sum_{m\neq n} \frac{b_m}{n-m}, \quad n \in Z,$$

is called the Hilbert transform of the sequence $\{b_n\}_{n\in Z}$.

M. Riesz (see [18, see also 10, 15]) proved that if $\{b_n\}_{n\in Z} \in l_p$, $p > 1$, then $\{\tilde{b}_n\}_{n\in Z} \in l_p$ and the inequality

$$\left\|\tilde{b}_n\right\|_p \leq C_p \|b_n\|_p \tag{1}$$

Communicated by Vladimir Bolotnikov.

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holds. Weighted analogues of (1) are investigated in the works [7–9, 13, 14, 16, 17, 21].

If $\{b_n\}_{n\in Z} \in l_1$, then the sequence $\{\tilde{b}_n\}_{n\in Z}$ belong to the class $\bigcap\limits_{p>1} l_p$, but it does not belong to the class l_1. In this case, R. Hunt, B. Muckenhoupt and R. Wheeden (see [14]) proved that the distribution function $\tilde{b}(\lambda) = \sum_{\left\{n\in Z:\, \left|\tilde{b}_n\right|>\lambda\right\}} 1$ of the Hilbert transform of the sequence $\{b_n\}_{n\in Z}$ satisfies the condition

$$\forall\lambda > 0: \ \left|\tilde{b}(\lambda)\right| \le \frac{C_0}{\lambda}\sum_{n\in Z}|b_n|, \tag{2}$$

where C_0 is an absolute constant. Note that for the sequence $\{b_n\}_{n\in Z} \in l_1$ the series $\sum_{n\in Z}\tilde{b}_n$ does not converge even in the sense of the principal value, i.e., in the sense

$$\sum_{n\in Z}\tilde{b}_n = \lim_{N\to\infty}\sum_{|n|\le N}\tilde{b}_n.$$

In the present paper, we study the asymptotic behavior of the distribution function $\tilde{b}(\lambda)$ of the Hilbert transform of a sequence $\{b_n\}_{n\in Z} \in l_1$ as $\lambda \to 0$ (Theorem 1). We introduce the concept of Q-summability of series and, using this notion, prove that the Hilbert transform of a sequence $\{b_n\}_{n\in Z} \in l_1$ is Q-summable and its Q-sum is zero (Theorem 2).

2 Asymptotic Behavior of the Distribution Function of the Discrete Hilbert Transform

Theorem 1 *Let* $\{b_n\}_{n\in Z} \in l_1$. *Then*

$$\lim_{\lambda\to 0+}\lambda\cdot\tilde{b}(\lambda) = 2\left|\sum_{n\in Z}b_n\right|, \tag{3}$$

where $\tilde{b}(\lambda) = \sum_{\left\{n\in Z:\, \left|\tilde{b}_n\right|>\lambda\right\}} 1$ *is the distribution function of the Hilbert transform of* $\{b_n\}_{n\in Z}$.

We first prove the auxiliary lemma.

Lemma 1 *Let* $\{b_n\}_{n\in Z} \in l_1$ and $\sum_{n\in Z} b_n = 0$. *Then*

$$\tilde{b}(\lambda) = o\left(\frac{1}{\lambda}\right), \ \ \lambda \to 0+. \tag{4}$$

Proof of Lemma 1 We first assume that a sequence $\{b_n\}_{n\in Z} \in l_1$ is concentrated on some finite interval $[-m, m]$, that is, $b_n = 0$ for $|n| > m$. For every $|k| < m$ and $|n| > 2m$, we have

$$|n-k| \geq |n| - |k| > |n| - m > |n| - \frac{|n|}{2} = \frac{|n|}{2}, \qquad \left| n - \frac{1}{2} \right| \geq |n| - \frac{1}{2} \geq \frac{|n|}{2}.$$

Therefore, in this case, for $|n| > 2m$ from the equality

$$\begin{aligned} \tilde{b}_n &= \sum_{k \neq n} \frac{b_k}{n-k} = \sum_{|k| \leq m} \frac{b_k}{n-k} = \sum_{|k| \leq m} \frac{b_k}{n-k} - \frac{1}{n-1/2} \sum_{k \in Z} b_k \\ &= \sum_{|k| \leq m} \frac{b_k}{n-k} - \frac{1}{n-1/2} \sum_{|k| \leq m} b_k = \sum_{|k| \leq m} \frac{k-1/2}{(n-k)(n-1/2)} b_k \end{aligned}$$

we obtain

$$\left|\tilde{b}_n\right| \leq \sum_{|k| \leq m} \frac{|k-1/2|}{|n-k||n-1/2|} b_k \leq \frac{4}{n^2} \sum_{|k| \leq m} |k-1/2||b_k|. \tag{5}$$

Denote

$$M_0 = \sum_{|k| \leq m} |k-1/2||b_k|.$$

Then it follows from (5) that

$$\begin{aligned} &\left\{n \in Z: \left|\tilde{b}_n\right| > \lambda\right\} \subset \{n \in Z: \ |n| \leq 2m\} \cup \left\{n \in Z \backslash [-2m, 2m]: \ \frac{4}{n^2} M_0 > \lambda\right\} \\ &= \{n \in Z: \ |n| \leq 2m\} \cup \left\{n \in Z \backslash [-2m, 2m]: \ |n| < 2\sqrt{M_0/\lambda}\right\}. \end{aligned}$$

Hence we have

$$\tilde{b}(\lambda) = \sum_{\left\{n \in Z: \left|\tilde{b}_n\right| > \lambda\right\}} 1 \leq \sum_{\{n \in Z: \ |n| \leq 2m\}} 1 + \sum_{\{n \in Z \backslash [-2m, 2m]: \ |n| < 2\sqrt{M_0/\lambda}\}} 1 \leq 4m + 4\sqrt{M_0/\lambda} + 2,$$

whence the asymptotic Eq. (4) follows.

Now let us consider the general case when a sequence $\{b_n\}_{n \in Z} \in l_1$ is concentrated on Z. It follows from the condition $\{b_n\}_{n \in Z} \in l_1$ that, for any $\varepsilon > 0$, there exist a number $m_\varepsilon \in N$ satisfying

$$\sum_{|n| > m_\varepsilon} |b_n| < \frac{\varepsilon}{8C_0},$$

where C_0 is the constant from (2). Setting

$$b'_n = \begin{cases} 0, & \text{for } |n| > m_\varepsilon \\ b_n - \frac{1}{2m_\varepsilon + 1} \sum\limits_{|k| \leq m_\varepsilon} b_k, & \text{for } |n| \leq m_\varepsilon \end{cases}, \quad b''_n = \begin{cases} b_n, & \text{for } |n| > m_\varepsilon \\ \frac{1}{2m_\varepsilon + 1} \sum\limits_{|k| \leq m_\varepsilon} b_k, & \text{for } |n| \leq m_\varepsilon \end{cases}$$

we have

$$b_n = b'_n + b''_n,$$

the sequence $\{b'_n\}_{n\in Z} \in l_1$ is concentrated on the finite interval $[-m_\varepsilon,\ m_\varepsilon]$,

$$\begin{aligned}\sum_{n\in Z} b'_n &= \sum_{|n|\le m_\varepsilon} b'_n = \sum_{|n|\le m_\varepsilon}\left[b_n - \frac{1}{2m_\varepsilon+1}\sum_{|k|\le m_\varepsilon} b_k\right]\\ &= \sum_{|n|\le m_\varepsilon} b_n - \frac{1}{2m_\varepsilon+1}(2m_\varepsilon+1)\sum_{|k|\le m_\varepsilon} b_k = 0;\end{aligned}$$

and the sequence $\{b''_n\}_{n\in Z} \in l_1$ satisfies the inequality

$$\begin{aligned}\sum_{n\in Z}|b''_n| &= \sum_{|n|>m_\varepsilon}|b''_n| + \sum_{|n|\le m_\varepsilon}|b''_n| = \sum_{|n|>m_\varepsilon}|b_n| + \sum_{|n|\le m_\varepsilon}\left|\frac{1}{2m_\varepsilon+1}\sum_{|k|\le m_\varepsilon} b_k\right|\\ &= \sum_{|n|>m_\varepsilon}|b_n| + \left|\sum_{|k|\le m_\varepsilon} b_k\right| = \sum_{|n|>m_\varepsilon}|b_n| + \left|\sum_{k\in Z} b_k - \sum_{|k|>m_\varepsilon} b_k\right|\\ &= \sum_{|n|>m_\varepsilon}|b_n| + \left|\sum_{|k|>m_\varepsilon} b_k\right| \le 2\sum_{|n|>m_\varepsilon}|b_n| < \frac{\varepsilon}{4C_0}.\end{aligned} \tag{6}$$

Since the sequence $\{b'_n\}_{n\in Z} \in l_1$ is concentrated on $[-m_\varepsilon,\ m_\varepsilon]$ and $\sum_{n\in Z} b'_n = 0$, then for the sequence $\{b'_n\}_{n\in Z} \in l_1$ Eq. (4) is satisfied, and therefore, there exists $\lambda(\varepsilon) > 0$ such that, for $0 < \lambda < \lambda(\varepsilon)$,

$$\lambda\tilde{b'}\left(\frac{\lambda}{2}\right) < \frac{\varepsilon}{2}, \tag{7}$$

where $\tilde{b'}(\lambda) = \sum_{\left\{n\in Z:\ \left|\tilde{b'}_n\right|>\lambda\right\}} 1$. On the other hand, from (2) and (6) it follows that

$$\lambda\tilde{b''}\left(\frac{\lambda}{2}\right) \le 2C_0\sum_{n\in Z}|b''_n| < \frac{\varepsilon}{2} \tag{8}$$

for any $\lambda > 0$, where $\tilde{b''}(\lambda) = \sum_{\left\{n\in Z:\ \left|\tilde{b''}_n\right|>\lambda\right\}} 1$.

Since $\tilde{b}_n = \tilde{b'}_n + \tilde{b''}_n$ for every $n \in Z$, we have, for every $\lambda > 0$,

$$\left\{n\in Z:\ \left|\tilde{b}_n\right| > \lambda\right\} \subset \left\{n\in Z:\ \left|\tilde{b}'_n\right| > \lambda/2\right\} \cup \left\{n\in Z:\ \left|\tilde{b}''_n\right| > \lambda/2\right\}. \tag{9}$$

For any $0 < \lambda < \lambda(\varepsilon)$ from inequalities (7), (8) and inclusion (9) we get

$$\tilde{b}(\lambda) = \sum_{\left\{n\in Z:\ \left|\tilde{b}_n\right|>\lambda\right\}} 1 \le \sum_{\left\{n\in Z:\ \left|\tilde{b}'_n\right|>\lambda/2\right\}} 1 + \sum_{\left\{n\in Z:\ \left|\tilde{b}''_n\right|>\lambda/2\right\}} 1 = \tilde{b}'(\lambda/2) + \tilde{b}''(\lambda/2) < \frac{\varepsilon}{\lambda}.$$

This shows that equality (4) is valid for all $\{b_n\}_{n\in Z} \in l_1$ satisfying the condition $\sum_{n\in Z} b_n = 0$. This completes the proof of Lemma 1.

Proof of Theorem 1 In the case $\sum_{n\in Z} b_n = 0$ the assertion of the theorem follows from Lemma 1. Let us consider the case $\sum_{n\in Z} b_n = \alpha \ne 0$. Puttig

$$b'_n = \begin{cases} b_n, & \text{for } n \ne 0 \\ b_0 - \alpha, & \text{for } n = 0 \end{cases}, \quad b''_n = \begin{cases} 0, & \text{for } n \ne 0 \\ \alpha, & \text{for } n = 0 \end{cases}$$

we have $b_n = b'_n + b''_n$ and $\sum_{n\in Z} b'_n = 0$. It follows from Lemma 1 that

$$\tilde{b'}(\lambda) = o\left(\frac{1}{\lambda}\right), \quad \lambda \to 0+. \tag{10}$$

Since $\tilde{b''}_n = \frac{\alpha}{n}$ for $n \ne 0$, $\tilde{b''}_0 = 0$, we have

$$\tilde{b}''(\lambda) = \sum_{\left\{n\in Z:\ \left|\tilde{b}''_n\right|>\lambda\right\}} 1 = \sum_{\{n\in Z\setminus\{0\}:\ |\alpha/n|>\lambda\}} 1 = \sum_{\{n\in Z\setminus\{0\}:\ |n|<\alpha/\lambda\}} 1 = 2[|\alpha/\lambda|], \tag{11}$$

where $[|\alpha/\lambda|]$ is the integer part of the number $|\alpha/\lambda|$.

Since $\tilde{b}_n = \tilde{b'}_n + \tilde{b''}_n$ for every $n \in Z$, we have, for any $0 < \varepsilon < 1$,

$$\left\{n \in Z:\ \left|\tilde{b}_n\right| > \lambda\right\} \subset \left\{n \in Z:\ \left|\tilde{b'}_n\right| > \varepsilon\lambda\right\} \cup \left\{n \in Z:\ \left|\tilde{b''}_n\right| > (1-\varepsilon)\lambda\right\}$$
$$\left\{n \in Z:\ \left|\tilde{b}_n\right| > \lambda\right\} \supset \left\{n \in Z:\ \left|\tilde{b''}_n\right| > (1+\varepsilon)\lambda\right\} \setminus \left\{n \in Z:\ \left|\tilde{b'}_n\right| > \varepsilon\lambda\right\}.$$

Hence

$$\begin{aligned}\tilde{b}(\lambda) &= \sum_{\left\{n\in Z:\ \left|\tilde{b}_n\right|>\lambda\right\}} 1 \le \sum_{\left\{n\in Z:\ \left|\tilde{b}'_n\right|>\varepsilon\lambda\right\}} 1 + \sum_{\left\{n\in Z:\ \left|\tilde{b}''_n\right|>(1-\varepsilon)\lambda\right\}} 1 \\ &= \tilde{b}'(\varepsilon\lambda) + \tilde{b}''((1-\varepsilon)\lambda), \\ \tilde{b}(\lambda) &= \sum_{\left\{n\in Z:\ \left|\tilde{b}_n\right|>\lambda\right\}} 1 \ge \sum_{\left\{n\in Z:\ \left|\tilde{b}''_n\right|>(1+\varepsilon)\lambda\right\}} 1 - \sum_{\left\{n\in Z:\ \left|\tilde{b}'_n\right|>\varepsilon\lambda\right\}} 1 \\ &= \tilde{b}''((1+\varepsilon)\lambda) - \tilde{b}'(\varepsilon\lambda).\end{aligned}$$

Now, using (10), (11),

$$\limsup_{\lambda \to 0+} \lambda \cdot \tilde{b}(\lambda) \leq \limsup_{\lambda \to 0+} \lambda \cdot \left[\tilde{b}'(\varepsilon\lambda) + \tilde{b}''((1-\varepsilon)\lambda) \right]$$
$$= \limsup_{\lambda \to 0+} \lambda \cdot 2 \left[\left| \frac{\alpha}{(1-\varepsilon)\lambda} \right| \right] = \frac{2|\alpha|}{1-\varepsilon}, \tag{12}$$

$$\liminf_{\lambda \to 0+} \lambda \cdot \tilde{b}(\lambda) \geq \liminf_{\lambda \to 0+} \lambda \cdot \left[\tilde{b}''((1+\varepsilon)\lambda) - \tilde{b}'(\varepsilon\lambda) \right]$$
$$= \liminf_{\lambda \to 0+} \lambda \cdot 2 \left[\left| \frac{\alpha}{(1+\varepsilon)\lambda} \right| \right] = \frac{2|\alpha|}{1+\varepsilon}. \tag{13}$$

Since ε is arbitrary, it follows from (12), (13) that

$$\liminf_{\lambda \to 0+} \lambda \cdot \tilde{b}(\lambda) = \limsup_{\lambda \to 0+} \lambda \cdot \tilde{b}(\lambda) = 2|\alpha|.$$

Hence (3) holds. This completes the proof of Theorem 1.

3 *Q*-Summability of Series and the Hilbert Transform

For a measurable complex function f on an interval $[a, b] \subset R$, we set

$$[f(x)]_n = [f(x)]^n = f(x) \ \text{ for } \ |f(x)| \leq n,$$
$$[f(x)]_n = n \cdot \operatorname{sgn} f(x), \quad [f(x)]^n = 0 \ \text{ for } \ |f(x)| > n, \ \ n \in N,$$

where $\operatorname{sgn} z = z/|z|$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$.

In 1929, Titchmarsh [22] introduced the notions of Q- and Q'-integrals of a function measurable on $[a, b]$.

Definition 1 If the finite limit $\lim_{n\to\infty} \int_a^b [f(x)]_n dx$ ($\lim_{n\to\infty} \int_a^b [f(x)]^n dx$, respectively) exists, then f is said to be *Q-integrable* (*Q'-integrable*, respectively) on $[a, b]$; that is, $f \in Q[a, b]$ ($f \in Q'[a, b]$). The value of this limit is referred to as the *Q-integral* (*Q'-integral*) of this function and is denoted by

$$(Q) \int_a^b f(x) dx \ \left((Q') \int_a^b f(x) dx \right).$$

As in Definition 1, one defines the Q- and Q'-integrals for a function measurable on the real axis R.

Given a measurable complex function f on the real axis R, we set

$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = f(x) \ \text{ for } \ \delta \leq |f(x)| \leq \lambda,$$

$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = 0 \text{ for } |f(x)| < \delta,$$
$$[f(x)]_{\delta,\lambda} = \lambda\, \mathrm{sgn} f(x), \quad [f(x)]^{\delta,\lambda} = 0 \text{ for } |f(x)| > \lambda, \quad 0 < \delta < \lambda.$$

Definition 2 If the finite limit $\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_R [f(x)]_{\delta,\lambda} dx$ ($\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_R [f(x)]^{\delta,\lambda} dx$, respectively) exists, then f is said to be *Q-integrable* (*Q'-integrable*, respectively) on R; that is, $f \in Q(R)$ ($f \in Q'(R)$). The value of this limit is referred to as the *Q-integral* (*Q'-integral*) of this function and is denoted by

$$(Q)\int_R f(x)dx \quad \left((Q')\int_R f(x)dx\right).$$

A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when dealing with diverse problems of function theory is the absence of the additivity property; that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the conditions

$$\delta\, m\{x \in R : \ |f(x)| > \delta\} = o(1), \quad \delta \to 0+, \tag{14}$$

$$\lambda\, m\{x \in R : \ |f(x)| > \lambda\} = o(1), \quad \lambda \to +\infty, \tag{15}$$

to the definition of Q-integrability (Q'-integrability) of a function f, then the Q-integral and Q'-integral coincide $\big(Q(R) = Q'(R)\big)$ and these integrals become additive.

Definition 3 If $f \in Q'(R)$ (or $f \in Q(R)$) and conditions (14) and (15) holds, then f is said to be A-integrable on R; that is, $f \in A(R)$. In this case, the limit $\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_R [f(x)]^{\delta,\lambda} dx$ (or the limit $\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_R [f(x)]_{\delta,\lambda} dx$) is denoted by

$$(A)\int_R f(x)dx.$$

Properties of Q- and Q'-integrals were investigated in [2, 3, 6, 11, 12, 22]; for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables we refer the reader to [1–6, 19, 20, 22, 23].

We need the following theorem proved in [3] and [4].

Theorem A [4, Theorem 4] Let ν be a finite complex measure on the real axis R. Then

$$(Q')\int_R (H\nu)(x)dx = 0,$$

where $(H\nu)(x) = \frac{1}{\pi}\int_R \frac{d\nu(t)}{x-t}$ is the Hilbert transform of the measure ν.

Definition 4 We denote by $M(R; C)$ the class of measurable complex-valued functions f on R for which the finite limits $\lim_{\lambda\to+\infty} \lambda m\{z \in R : |f(z)| > \lambda\}$ and $\lim_{\delta\to 0+} \delta m\{z \in R : |f(z)| > \delta\}$ exist.

Remark 1 Note that the Hilbert transform of a finite complex measure belong to the class of functions $M(R; C)$ (see [4]).

Theorem B [3, Theorem 2.3] If a function $f \in M(R; C)$ is Q'-integrable on R and a function g is A-integrable on R, then their sum $f + g \in M(R; C)$ is Q'-integrable on R, and

$$(Q')\int_R [f(x) + g(x)]dx = (Q')\int_R f(x)dx + (A)\int_R g(x)dx.$$

Similar to the definition of the Q-integral, we define the Q-sum of series. Let $\{a_n\}_{n\in Z}$ be a sequence of complex numbers.

Definition 5 If the finite limit $\lim_{\lambda\to 0+} \sum_{\{n\in Z:\, |a_n|\geq\lambda\}} a_n$ exists, then the series $\sum_{n\in Z} a_n$ is said to be Q-summable, and the value of this limit is referred to as the Q-sum of this series and is denoted by

$$(Q)\sum_{n\in Z} a_n.$$

Q-summable series does not enjoy the additivity property; that is, the Q-summability of two series does not imply the Q-summability of their sum. If one adds the condition

$$\sum_{\{n\in Z:\, |a_n|>\lambda\}} 1 = o\left(\frac{1}{\lambda}\right), \quad \lambda \to 0+ \tag{16}$$

to the definition of Q-summability of a series $\sum_{n\in Z} a_n$, then the Q-sum become additive.

Definition 6 If a series $\sum_{n\in Z} a_n$ is Q-summable and condition (16) holds, then the series $\sum_{n\in Z} a_n$ is said to be A-summable, and the limit $\lim_{\lambda\to 0+} \sum_{\{n\in Z:\, |a_n|\geq\lambda\}} a_n$ is denoted in this case by $(A)\sum_{n\in Z} a_n$.

Theorem 2 *Let $\{b_n\}_{n\in Z} \in l_1$. Then the series $\sum_{n\in Z} \tilde{b}_n$ is Q-summable and the equation*

$$(Q)\sum_{n\in Z} \tilde{b}_n = 0 \tag{17}$$

holds.

Proof of Theorem 2 Define the function $f(x)$ to be $2\pi b_n$ for $x \in [n - 1/4,\ n + 1/4]$, $n \in Z$ and 0 elsewhere, the function $F(x)$ to be $\tilde{b}_n$ for $x \in [n - 1/2,\ n + 1/2)$, $n \in Z$ and

$$G(x) = (Hf)(x) - F(x).$$

We first show that $G_1(x) \in L_1(R)$.

For every $x \in [n - 1/2,\ n + 1/2)$, $x \neq n \pm 1/4$ we have

$$\begin{aligned} G(x) &= \frac{1}{\pi}\int_R \frac{f(t)}{x-t}dt - \tilde{b}_n = \frac{1}{\pi}\sum_{m\in Z}\int_{m-1/4}^{m+1/4}\frac{2\pi b_m}{x-t}dt - \sum_{m\neq n}\frac{b_m}{n-m} \\ &= \left(\sum_{m\neq n} 2b_m \int_{m-1/4}^{m+1/4}\frac{dt}{x-t} + 2b_n\int_{n-1/4}^{n+1/4}\frac{dt}{x-t}\right) - \sum_{m\neq n}\frac{b_m}{n-m} \\ &= \sum_{m\neq n} 2b_m\left(\int_{m-1/4}^{m+1/4}\left(\frac{1}{x-t} - \frac{1}{n-m}\right)dt\right) + 2b_n\int_{n-1/4}^{n+1/4}\frac{dt}{x-t} = G_1(x) + G_2(x). \end{aligned} \tag{18}$$

Let $m \neq n$. Then, for every $x \in [n - 1/2,\ n + 1/2)$ and $t \in [m - 1/4,\ m + 1/4]$, since

$$|x-n| \le 1/2, \quad |m-t| \le 1/4, \quad |x-t| \ge |n-m| - |x-n| - |m-t| \ge |n-m| - 3/4$$

we get

$$\left|\frac{1}{x-t} - \frac{1}{n-m}\right| = \frac{|n-x+t-m|}{|x-t|\cdot|n-m|} \le \frac{1/2+1/4}{|n-m|\cdot(|n-m|-3/4)} = \frac{3}{|n-m|\cdot(4|n-m|-3)}. \tag{19}$$

Therefore, for every $x \in [n - 1/2,\ n + 1/2)$,

$$|G_1(x)| \le \sum_{m\neq n} 2|b_m|\cdot\int_{m-1/4}^{m+1/4}\left|\frac{1}{x-t} - \frac{1}{n-m}\right|dt \le \sum_{m\neq n} 3|b_m|\cdot\frac{1}{|n-m|\cdot(4|n-m|-3)}. \tag{20}$$

From inequality (20) it follows that

$$\int_R |G_1(x)|dx = \sum_{n\in Z}\int_{n-1/2}^{n+1/2}|G_1(x)|dx \le \sum_{n\in Z}\sum_{m\neq n} 3|b_m|\cdot\frac{1}{|n-m|\cdot(4|n-m|-3)}. \tag{21}$$

Since for every $m \in Z$ the series $\sum_{n\neq m}\frac{1}{|n-m|\cdot(4|n-m|-3)} = \sum_{k\neq 0}\frac{1}{|k|\cdot(4|k|-3)}$ is convergent, we have from (21)

$$\int_R |G_1(x)|dx \le \sum_{m\in Z} 3|b_m| \sum_{n\neq m} \frac{1}{|n-m|\cdot(4|n-m|-3)}$$

$$= 3\sum_{m\in Z}|b_m|\cdot\sum_{k\neq 0}\frac{1}{|k|\cdot(4|k|-3)} = 3\sum_{k\neq 0}\frac{1}{|k|\cdot(4|k|-3)}\cdot\sum_{m\in Z}|b_m|$$

and, therefore, $G_1(x) \in L_1(R)$.

Let us show that $G_2(x) \in L_1(R)$.

For every $n \in Z$ we subdivide the set $[n-1/2,\ n+1/2)\backslash\{n-1/4;\ n+1/4\}$ into four parts: $[n-1/2,\ n-1/4)$, $(n-1/4,\ n]$, $(n,\ n+1/4)$, $(n+1/4,\ n+1/2)$.

If $x \in [n-1/2,\ n-1/4)$, then

$$G_2(x) = 2b_n \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n[-\ln(n+1/4-x)+\ln(n-1/4-x)];$$

For every $x \in R$ and $\delta > 0$, the equality

$$\text{v.p.}\int_{x-\delta}^{x+\delta}\frac{dt}{x-t} = \text{v.p.}\int_{-\delta}^{\delta}\frac{du}{-u} = 0$$

holds. Therefore, if $x \in (n-1/4,\ n]$, then

$$G_2(x) = 2b_n\text{v.p.}\int_{n-1/4}^{n+1/4}\frac{dt}{x-t} = 2b_n\left(\text{v.p.}\int_{x-(x-n+1/4)}^{x+(x-n+1/4)}\frac{dt}{x-t} + \int_{x+(x-n+1/4)}^{n+1/4}\frac{dt}{x-t}\right)$$

$$= 2b_n\int_{2x-n+1/4}^{n+1/4}\frac{dt}{x-t} = 2b_n[-\ln(n+1/4-x)+\ln(x-n+1/4)]$$

if $x \in (n,\ n+1/4)$, then

$$G_2(x) = 2b_n\text{v.p.}\int_{n-1/4}^{n+1/4}\frac{dt}{x-t} = 2b_n\left(\text{v.p.}\int_{n-1/4}^{x-(n+1/4-x)}\frac{dt}{x-t} + \int_{x-(n+1/4-x)}^{x+(n+1/4-x)}\frac{dt}{x-t}\right)$$

$$= 2b_n\int_{n-1/4}^{2x-n-1/4}\frac{dt}{x-t} = 2b_n[-\ln(n+1/4-x)+\ln(x-n+1/4)];$$

if $x \in (n + 1/4,\ n + 1/2)$, then

$$G_2(x) = 2b_n \int\limits_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n[-\ln(x-n-1/4) + \ln(x-n+1/4)].$$

This shows that for every $x \in [n-1/2,\ n+1/2)$, $x \neq n \pm 1/4$ we have

$$G_2(x) = 2b_n[-\ln|x-n-1/4| + \ln|x-n+1/4|] \tag{22}$$

and, therefore,

$$|G_2(x)| \leq |2b_n| \left[\ln \frac{1}{|x-n-1/4|} + \ln \frac{1}{|x-n+1/4|} \right].$$

Now, for every $n \in Z$,

$$\int\limits_{n-1/2}^{n+1/2} |G_2(x)| dx \leq |2b_n| \cdot M_1,$$

where $M_1 = \int_{n-1/2}^{n+1/2} \left[\ln \frac{1}{|x-n-1/4|} + \ln \frac{1}{|x-n+1/4|} \right] dx = \int_{-1/2}^{1/2} \left[\ln \frac{1}{|u-1/4|} + \ln \frac{1}{|u+1/4|} \right] du$. Therefore,

$$\int_R |G_2(x)| dx = \sum_{n\in Z} \int_{n-1/2}^{n+1/2} |G_2(x)| dx \leq 2M_1 \sum_{n \in Z} |b_n|.$$

It follows that $G_2(x) \in L_1(R)$, and hence $G(x) \in L_1(R)$.

Now we prove that the series $\sum_{n\in Z} \tilde{b}_n$ is Q-summable and Eq. (17) holds.

Since $F(x) = (Hf)(x) - G(x)$, $Hf \in M(R;\ C)$ (see: Remark 1) and $G(x) \in L_1(R)$, it follows from Theorems A and B that the function $F(x)$ is Q'-integrable on R, and moreover,

$$(Q') \int\limits_R F(x) dx = (Q') \int\limits_R (Hf)(x) dx - \int\limits_R G(x) dx = -\int\limits_R G(x) dx \tag{23}$$

The function $F(x)$ is bounded and by definition for every $\lambda > 0$

$$\{x \in R : |F(x)| > \lambda\} = \bigcup_{n \in Z} \left\{ x \in [n-1/2,\ n+1/2) : \left| \tilde{b}_n \right| > \lambda \right\} = \bigcup_{\left\{n:\left|\tilde{b}_n\right| > \lambda\right\}} [n-1/2,\ n+1/2),$$

hence

$$(Q')\int_R F(x)dx = \lim_{\substack{\lambda\to 0+\\ \delta\to+\infty}} \int_{\{x\in R:\,\lambda<|F(x)|<\delta\}} F(x)dx = \lim_{\lambda\to 0+} \int_{\{x\in R:\,|F(x)|>\lambda\}} F(x)dx$$

$$= \lim_{\lambda\to 0+} \sum_{\left\{n:\,\left|\tilde{b}_n\right|>\lambda\right\}} \int_{n-1/2}^{n+1/2} \tilde{b}_n dx = \lim_{\lambda\to 0+} \sum_{\left\{n:\,\left|\tilde{b}_n\right|>\lambda\right\}} \tilde{b}_n = (Q)\sum_{n\in Z} \tilde{b}_n. \tag{24}$$

It follows from (23) and (24) that the series $\sum_{n\in Z}\tilde{b}_n$ is Q-summable and the equation

$$(Q)\sum_{n\in Z}\tilde{b}_n = -\int_R G(x)dx. \tag{25}$$

holds. For every $n \in Z$ it follows from (22) that

$$\int_{n-1/2}^{n+1/2} G_2(x)dx = 2b_n\left[-\int_{n-1/2}^{n+1/2} \ln|x-n-1/4|dx + \int_{n-1/2}^{n+1/2} \ln|x-n+1/4|dx\right]$$

$$= 2b_n\left[-\int_{-1/2}^{1/2} \ln|u-1/4|du + \int_{-1/2}^{1/2} \ln|u+1/4|du\right]$$

$$= 2b_n\left[\int_{-1/2}^{1/2} \ln|u-1/4|d(-u) + \int_{-1/2}^{1/2} \ln|u+1/4|du\right]$$

$$= 2b_n\left[-\int_{-1/2}^{1/2} \ln|-z-1/4|dz + \int_{-1/2}^{1/2} \ln|u+1/4|du\right] = 0.$$

Therefore,

$$\int_R G_2(x)dx = \sum_{n\in Z}\int_{n-1/2}^{n+1/2} G_2(x)dx = 0. \tag{26}$$

By (19) for every $m \in Z$ and $t \in [m-1/4, m+1/4]$ the series $\sum_{n\neq m}\int_{n-1/2}^{n+1/2}\left|\frac{1}{x-t}-\frac{1}{n-m}\right|dx$ is convergent. Hence

$$\sum_{n\neq m}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx$$
$$=\lim_{p\to\infty}\left(\sum_{n=m-p}^{m-1}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx+\sum_{n=m+1}^{m+p}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx\right).\quad(27)$$

Since for every $m\in Z$ and $p\in N$

$$\sum_{n=m-p}^{m-1}\frac{1}{n-m}+\sum_{n=m+1}^{m+p}\frac{1}{n-m}=\left(-\frac{1}{p}-\frac{1}{p-1}-\cdots-\frac{1}{2}-1\right)$$
$$+\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}+\frac{1}{p}\right)=0,$$

it follows from (27) that

$$\sum_{n\neq m}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx$$
$$=\lim_{p\to\infty}\left(\sum_{n=m-p}^{m-1}\int_{n-1/2}^{n+1/2}\frac{1}{x-t}dx+\sum_{n=m+1}^{m+p}\int_{n-1/2}^{n+1/2}\frac{1}{x-t}dx\right)=\lim_{p\to\infty}\left(\int_{m-p-1/2}^{m-1/2}\frac{1}{x-t}dx+\int_{m+1/2}^{m+p+1/2}\frac{1}{x-t}dx\right)$$
$$=\lim_{p\to\infty}\left[\ln(t-m+1/2)-\ln(t-m+p+1/2)+\ln(m+p+1/2-t)-\ln(m+1/2-t)\right]$$
$$=\ln(t-m+1/2)-\ln(m+1/2-t).$$

Therefore, for every $m\in Z$,

$$\int_{m-1/4}^{m+1/4}\left[\sum_{n\neq m}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx\right]dt=\int_{m-1/4}^{m+1/4}[\ln(t-m+1/2)-\ln(m+1/2-t)]dt$$
$$=\int_{-1/4}^{1/4}[\ln(u+1/2)-\ln(1/2-u)]du=\int_{-1/4}^{1/4}\ln(u+1/2)du+\int_{-1/4}^{1/4}\ln(1/2-u)d(-u)$$
$$=\int_{-1/4}^{1/4}\ln(u+1/2)du-\int_{-1/4}^{1/4}\ln(1/2+z)dz=0.$$

Now it follows from Fubini's theorem that

$$\int_R G_1(x)dx=\sum_{n\in Z}\int_{n-1/2}^{n+1/2}G_1(x)dx=\sum_{n\in Z}\int_{n-1/2}^{n+1/2}\left[\sum_{m\neq n}2b_m\int_{m-1/4}^{m+1/4}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dt\right]dx$$
$$=\sum_{m\in Z}2b_m\left[\int_{m-1/4}^{m+1/4}\left[\sum_{n\neq m}\int_{n-1/2}^{n+1/2}\left(\frac{1}{x-t}-\frac{1}{n-m}\right)dx\right]dt\right]=0.\quad(28)$$

Now from Eqs. (18), (25), (26) and (28) we finally obtain (17). This completes the proof of theorem 2.

Acknowledgements The authors are very grateful to the referee for many useful suggestions and comments that improved the original manuscript. The research of the first author was supported by the Science Development Foundation under the President of the Republic of Azerbaijan (Grant No. EIF/MQM/Elm-Tehsil-1-2016-1(26)-71/08/01).

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