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TWO-WEIGHTED INEQUALITIES FOR SOME SUBLINEAR OPERATORS IN LEBESGUE SPACES

Abstract

In this paper, the author establishes the boundedness in weighted L_p spaces on $\mathbb{R}^n$ a large class of sublinear operators generated by Calderon-Zygmund operators. The conditions of these theorems are satisfied by many important operators in analysis and these operators satisfy only some weak conditions on the size of operators and are known to be bounded in the unweighted case. Sufficient conditions on weighted functions ω and ω_1 are given so that certain sublinear operator is bounded from the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^n)$ into $L_{p,\omega_1}(\mathbb{R}^n)$.

In the paper, we will prove the boundedness of some sublinear operators on the weighted L_p spaces.

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x', x'') = (x_1, ..., x_n)$, $x' = (x_1, ..., x_m)$, $x'' = (x_{m+1}, ..., x_n)$, $1 \le m \le n$, $|x'|^2 = \sum_{i=1}^n x_i^2$, An almost everywhere positive and locally integrable function $\omega : \mathbb{R}^n \to R$, will be called a weight. We shall denote by $L_{p,\omega}(\mathbb{R}^n)$ the set of all measurable function f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)} \equiv \|f\|_{p,\omega;\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p}, \qquad 1 \le p < \infty,$$

is finite.

Definition 1. *A function K defined on $\mathbb{R}^n \setminus \{0\}$, is said to be a Calderon-Zygmund (CZ) kernel in the space $\mathbb{R}^n$ if*

i) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

ii) $K(rx) = r^{-n} K(x)$ for each $r > 0$, $x \in \mathbb{R}^n \setminus \{0\}$;

iii) $\int_\Sigma K(x) d\sigma_x = 0$, where $d\sigma$ is the element of area of the $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$;

A sufficient condition for Calderon–Zygmund operator $T : L_{p,\omega}(\mathbb{R}^n) \to L_{p,\omega_1}(\mathbb{R}^n)$ was found by N.Fuji [6], however the condition he introduced is not easy to check for given weights. Recently Guliyev [7] and Edmunds and Kokilashvili [14] found new sufficient conditions easily verifiable for Calderon–Zygmund operator $T : L_{p,\omega}(\mathbb{R}^n) \to L_{p,\omega_1}(\mathbb{R}^n)$, whenever $\omega(\cdot)$ and $\omega_1(\cdot)$ are radial monotone weights. In the paper Y. Rakotondratsimba [16] was proved the following theorem

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Theorem 1. *Let $p \in (1,\infty)$, T be a Calderon–Zygmund operator. Moreover, let $\omega(x)$,$\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and the following three conditions be satisfied:*
there exists a constant $b>0$ such that

$$\sup_{|x|/4<|y|<4|x|} \omega_1(y) \le b\,\omega(x) \quad \textit{for a.e. } x\in\mathbb{R}^n,$$

$$\sup_{r>0}\left(\int_{|x|>2r}\omega_1(x)|x|^{-np}dx\right)\left(\int_{|x|<r}\omega^{1-p'}(x)dx\right)^{p-1}<\infty,$$

$$\sup_{r>0}\left(\int_{|x|<r}\omega_1(x)dx\right)\left(\int_{|x|>2r}\omega^{1-p'}(x)|x|^{-np'}dx\right)^{p-1}<\infty.$$

Then there exists a constant c, independent of f, such that for all $f\in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}|Tf(x)|^p\omega_1(x)dx \le c\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

Moreover, condition (a) can be replaced by the condition
there exists a constant $b>0$ such that

$$\omega_1(x)\left(\sup_{|x|/4<|y|<4|x|}\frac{1}{\omega(y)}\right)\le b \quad \textit{for a.e. } x\in\mathbb{R}^n.$$

We say that a locally integrable function $\omega\ :\mathbb{R}^n\to(0,\infty)$ satisfies Muckenhoupt's condition $A_p=A_p(\mathbb{R}^n)$ (briefly, $\omega\in A_p$), $1<p<\infty$, if there is a constant $C=C(\omega,p)$ such that for any ball $B\subset\mathbb{R}^n$

$$\left(|B|^{-1}\int_B\omega(x)dx\right)\left(|B|^{-1}\int_B\omega^{1-p'}(x)dx\right)^{p-1}\le C, \quad \frac{1}{p}+\frac{1}{p'}=1,$$

where the second factor on the left is replaced by ess sup $\{\omega^{-1}(x): x\in B\}$ if $p=1$.

First, we establish the boundedness in weighted L_p spaces for a large class of sublinear operators.

Theorem 2. *Let $p\in(1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ such that, for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$*

$$|Tf(x)|\le c_0\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy, \tag{1}$$

where c_0 is independent of f and x.

Moreover, let $\omega(x)=u(x)\phi(x)$, $\omega_1(x)=u_1(x)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x)\in A_p(\mathbb{R}^n)$ and the following three conditions be satisfied:

(a) there exists a constant $b>0$ such that

$$\sup_{|x|/4<|y|<4|x|} u_1(y)\le b\,u(x) \quad \textit{for a.e. } x\in\mathbb{R}^n,$$

$$(b) \qquad \mathcal{A}\equiv\sup_{r>0}\left(\int_{|x|>2r}\omega_1(x)|x|^{-np}dx\right)\left(\int_{|x|<r}\omega^{1-p'}(x)dx\right)^{p-1}<\infty,$$

$$(c) \qquad \mathcal{B} \equiv \sup_{r>0} \left(\int_{|x|<r} \omega_1(x)dx \right) \left(\int_{|x|>2r} \omega^{1-p'}(x)|x|^{-np'}dx \right)^{p-1} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(x)dx \le c \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx. \tag{2}$$

Moreover, condition (a) *can be replaced by the condition*

(a') *there exist* $b > 0$ *such that*

$$u_1(x) \left(\sup_{|x|/4 \le |y| \le 4|x|} \frac{1}{u(y)} \right) \le b \quad \textit{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $k \in Z$ we define $E_k = \{x \in R^n : 2^k < |x| \le 2^{k+1}\}$, $E_{k,1} = \{x \in R^n : |x| \le 2^{k-1}\}$, $E_{k,2} = \{x \in R^n : 2^{k-1} < |x| \le 2^{k+2}\}$, $E_{k,3} = \{x \in R^n : |x| > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k \in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^n)$, we write

$$|Tf(x)| = \sum_{k \in Z} |Tf(x)|\chi_{E_k}(x) \le \sum_{k \in Z} |Tf_{k,1}(x)|\,\chi_{E_k}(x) +$$

$$+ \sum_{k \in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(x) + \sum_{k \in Z} |Tf_{k,3}(x)|\,\chi_{E_k}(x) \equiv T_1 f(x) + T_2 f(x) + T_3 f(x), \tag{3}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$.

First we shall estimate $\|T_1 f\|_{L_{p,\omega_1}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $\rho|(y) \le 2^{k-1} \le |x|/2$. Moreover, $E_k \cap supp f_{k,1} = \emptyset$ and $|x - y| \ge |x|/2$. Hence by (1)

$$T_1 f(x) \le c_0 \sum_{k \in Z} \left(\int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{|x-y|^n} dy \right) \chi_{E_k} \le$$

$$\le c_0 \int_{|y| \le |x|/2} |x-y|^{-n} |f(y)|dy \le 2^n c_0 |x|^{-n} \int_{|y| \le |x|/2} |f(y)|dy$$

for any $x \in E_k$. Hence we have

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x)dx \le (2^n c_0)^p \int_{\mathbb{R}^n} \left(\int_{|y|<|x|/2} |f(y)|dy \right)^p |x|^{-np} \omega_1(x)dx.$$

Since $\mathcal{A} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n} \omega_1(x)|x|^{-np} \left(\int_{|y|<|x|/2} |f(y)|dy \right)^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx$$

holds and $C \le c'\mathcal{A}$, where c' depends only on n, a and p. In fact the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [13]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x)dx \le c_1 \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx. \tag{4}$$

where c_1 is independent of f.

Next we estimate $\|T_3 f\|_{L_{p,\omega_1}}$. As is easy to verify, for $x \in E_k$, $y \in E_{k,3}$ we have $|y| > 2|x|$ and $|x-y| \ge |y|/2$. Since $E_k \cap supp f_{k,3} = \emptyset$, for $x \in E_k$ by (1) we obtain

$$T_3 f(x) \le c_0 \int_{|y|>2|x|} \frac{|f(y)|}{|x-y|^n} dy \le 2^n c_0 \int_{|y|>2|x|} |f(y)||y|^{-n} dy.$$

Hence we have

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x) dx \le (2^n c_0)^p \int_{\mathbb{R}^n} \left(\int_{|y|>2|x|} |f(y)||y|^{-n} dy \right)^p \omega_1(x) dx.$$

Since $\mathcal{B} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n} \omega_1(x) \left(\int_{|y|>2|x|} |f(y)||y|^{-n} dy \right)^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

holds and $C \le c'\mathcal{B}$, where c' depends only on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [13]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x) dx \le c_2 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{5}$$

where c_2 is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_{p,\phi}(\mathbb{R}^n)$ boundedness of T and condition (a) we have

$$\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x) dx = \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |T f_{k,2}(x)| \, \chi_{E_k}(x) \right)^p \omega_1(x) dx =$$

$$= \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |T f_{k,2}(x)|^p \, \chi_{E_k}(x) \right) \omega_1(x) dx = \sum_{k\in Z} \int_{E_k} |T f_{k,2}(x)|^p \, u_1(x) \phi(x) dx \le$$

$$\le \sum_{k\in Z} \sup_{x\in E_k} u_1(x) \int_{\mathbb{R}^n} |T f_{k,2}(x)|^p \, \phi(x) dx \le \|T\|_\phi^p \sum_{k\in Z} \sup_{x\in E_k} u_1(x) \int_{\mathbb{R}^n} |f_{k,2}(x)|^p \, \phi(x) dx =$$

$$= \|T\|_\phi^p \sum_{k\in Z} \sup_{y\in E_k} u_1(y) \int_{E_{k,2}} |f(x)|^p \phi(x) dx,$$

where $\|T\|_\phi \equiv \|T\|_{L_{p,\phi}(\mathbb{R}^n)\to L_{p,\phi}(\mathbb{R}^n)}$. Since, for $x \in E_{k,2}$, $2^{k-1} < |x| \le 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} u_1(y) = \sup_{2^{k-1}<|y|\le 2^{k+2}} u_1(y) \le \sup_{|x|/4<|y|\le 4|x|} u_1(y) \le b\, u(x)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x) dx \le$$

$$\le \|T\|_\phi^p b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p u(x) \phi(x) dx \le c_3 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{6}$$

where $c_3 = 3\|T\|_\phi^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (3), (4), (5), (6) imply (2) which completes the proof.

Analoguoslu proved the following weak variant Theorem 1.

Theorem 3. *Let $p \in [1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let $\omega(x) = u(x)\phi(x)$, $\omega_1(x) = u_1(x)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$ and conditions (a), (b), (c) be satisfied.*

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\{x\in\mathbb{R}^n\,:\,|Tf(x)|>\lambda\}} \omega_1(x)dx \le \frac{c}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx. \tag{7}$$

Let K is a Calderon–Zygmund kernel and T be the corresponding integral operator

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^n} K(x-y)f(y)dy.$$

Then T satisfies the condition (1). See [15] for details. Thus, we have

Corollary 1. *Let $p \in (1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(x) = u(x)\phi(x)$, $\omega_1(x) = u_1(x)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$ and conditions (a), (b), (c) be satisfied. Then inequality (2) is valid.*

Corollary 2. *Let $p \in [1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(x) = u(x)\phi(x)$, $\omega_1(x) = u_1(x)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$ and conditions (a), (b), (c) be satisfied. Then inequality (7) is valid.*

Remark 1. *Note that, Corollary 1 were proved in [16] and for singular integral operators, defined on homogeneous groups in [12], [8] (see also [10], [2], [3], [7]).*

Theorem 4. *Let $p \in (1,\infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0,\infty)$, $u_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (b).*

Then inequality (2) is valid.

Proof. Suppose that $f \in L_{p,\omega}(\mathbb{R}^n)$ and ω_1 are positive increasing functions on $(0,\infty)$ and $(\omega(|x|), \omega_1(|x|))$ satisfied the conditions (a), (b).

Without loss of generality we can suppose that u_1 may be represented by

$$u_1(t) = u_1(0+) + \int_0^t \psi(\lambda)d\lambda,$$

where $u_1(0+) = \lim_{t\to 0} u_1(t)$ and $u_1(t) \ge 0$ on $(0,\infty)$. In fact there exists a seqence of increasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [10], [2], [7], [3], [9] for details).

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p\omega_1(x)dx = u_1(0+)\int_{\mathbb{R}^n} |Tf(x)|^p\phi(x)dx+$$

$$+\int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_0^{|x|} \psi(\lambda)d\lambda\right) \phi(x)dx = J_1 + J_2.$$

If $u_1(0+) = 0$, then $J_1 = 0$. If $u_1(0+) \neq 0$ by the boundedness of T in $L_{p,\phi}(\mathbb{R}^n)$, $\phi \in A_p$ thanks to (a)

$$J_1 \leq \|T\|_\phi^p u_1(0+) \int_{\mathbb{R}^n} |f(x)|^p \phi(x)dx \leq$$

$$\leq \|T\|_\phi^p \int_{\mathbb{R}^n} |f(x)|^p u_1(|x|)\phi(x)dx \leq b\,\|T\|_\phi^p \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{|x|>\lambda} |Tf(x)|^p \phi(x)dx\right) d\lambda \leq$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{|x|>\lambda} |T(f\chi_{\{|x|>\lambda/2\}})(x)|^p \phi(x)dx +\right.$$

$$\left. + \int_{|x|>\lambda} |T(f\chi_{\{|x|\leq\lambda/2\}})(x)|^p \phi(x)dx\right) d\lambda = J_{21} + J_{22}.$$

Using the boundedness of T in $L_{p,\phi}(\mathbb{R}^n)$ and condition (a) we have

$$J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left(\int_{|y|>\lambda/2} |f(y)|^p \phi(y)dy\right) dt =$$

$$= \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \left(\int_0^{2|y|} \psi(\lambda)d\lambda\right) \phi(y)dy \leq$$

$$\leq \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p u_1(2|y|)\phi(y)dy \leq b\,\|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \omega(y)dy.$$

Let us estimate J_{22}. For $|x| > \lambda$ and $|y| \leq \lambda/2$ we have $|x|/2 \leq |x-y| \leq 3|x|/2$, and so

$$J_{22} \leq c_4 \int_0^\infty \psi(\lambda) \left(\int_{|x|>\lambda} \left(\int_{|y|\leq 2\lambda} \frac{|f(y)|}{|x-y|^n} dy\right)^p \phi(x)dx\right) d\lambda \leq$$

$$\leq c_5 \int_0^\infty \psi(\lambda) \left(\int_{|x|>\lambda} \left(\int_{|y|\leq 2\lambda} |f(y)|dy\right)^p |x|^{-np} \phi(x)dx\right) d\lambda =$$

$$= c_6 \int_0^\infty \psi(\lambda)\lambda^{-np+n} \left(\int_{|y|\leq\lambda/2} |f(y)|dy\right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{-np+n} \left(\int_{|y|\leq\lambda/2} |f(y)|dy\right)^p d\lambda \leq C \int_{\mathbb{R}^n} |f(y)|^p \omega(|y|)dy$$

for $p \in (1, \infty)$ is characterized by the condition $C \le c' \mathcal{A}'$ (see [4], [11]), where

$$\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^{\infty} \psi(t) t^{-np+n} d\tau \right) \left(\int_0^{\tau} \omega^{1-p'}(t) dt \right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^{\infty} \psi(\tau)\tau^{-np+n} d\tau = n(p-1) \int_{2t}^{\infty} \psi(\tau) d\tau \int_{\tau}^{\infty} \lambda^{n-1-np} d\lambda =$$
$$= n(p-1) \int_{2t}^{\infty} \lambda^{n-1-np} d\lambda \int_{2t}^{\lambda} \psi(\tau) d\tau \le$$
$$\le n(p-1) \int_{2t}^{\infty} \lambda^{n-1-np} \omega_1(\lambda) d\lambda = c_{10} \int_{|x|>2r} \omega_1(|x|)|x|^{-np} dx.$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \le c_{10}\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c_7 \int_{\mathbb{R}^n} |f(x)|^p \omega(|x|) dx.$$

Combining the estimates of J_1 and J_2, we get (2) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau) d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (2). The theorem is proved.

Theorem 5. *Let $p \in [1, \infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying* (1)*. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0, \infty)$, $u_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (b).*

Then inequality (7) *is valid.*

Corollary 3. *Let $p \in (1, \infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0, \infty)$, $u_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (b). Then inequality* (2) *is valid.*

Corollary 4. *Let $p \in [1, \infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0, \infty)$, $u_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (b). Then inequality* (7) *is valid.*

Theorem 6. *Let $p \in (1, \infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying* (1)*. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0, \infty)$, $u_1(t)$ be a positive decreasing function on $(0, \infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (c). Then inequality* (2) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$(see [7], [9] for details).

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(|x|)dx = \omega_1(+\infty) \int_{\mathbb{R}^n} |Tf(x)|^p dx +$$

$$+ \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_{|x|}^\infty \psi(\tau)d\tau \right) dx = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a) we have

$$J_1 \leq \|T\| \omega_1(+\infty) \int_{\mathbb{R}^n} |f(x)|^p dx \leq$$

$$\leq \|T\| \int_{\mathbb{R}^n} |f(x)|^p \omega_1(|x|)dx \leq b\,\|T\| \int_{\mathbb{R}^n} |f(x)|^p \omega(|x|)dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{|x|<\lambda} |Tf(x)|^p dx \right) d\lambda \leq$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{|x|<\lambda} |T(f\chi_{\{|x|<2\lambda\}})(x)|^p dx + \right.$$

$$\left. + \int_{|x|<\lambda} |T(f\chi_{\{|x|\geq 2\lambda\}})(x)|^p dx \right) d\lambda = J_{21} + J_{22}.$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a') we obtain

$$J_{21} \leq \|T\| \int_0^\infty \psi(t) \left(\int_{|y|<2\lambda} |f(y)|^p dy \right) dt = \|T\| \int_{\mathbb{R}^n} |f(y)|^p \left(\int_{|y|/2}^\infty \psi(\lambda)d\lambda \right) dy \leq$$

$$\leq \|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega_1(|y|/2)dy \leq b\,\|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega(|y|)dy.$$

Let us estimate J_{22}. For $|x| < \lambda$ and $|y| \geq 2\lambda$ we have $|y|/2 \leq |x-y| \leq 3|y|/2$, and so

$$J_{22} \leq c_8 \int_0^\infty \psi(\lambda) \left(\int_{|x|<\lambda} \left(\int_{|y|\geq 2\lambda} \frac{|f(y)|}{|x-y|^n} dy \right)^p dx \right) d\lambda \leq$$

$$\leq 2^n c_8 \int_0^\infty \psi(\lambda) \left(\int_{|x|<\lambda} \left(\int_{|y|\geq 2\lambda} |y|^{-n} |f(y)| dy \right)^p dx \right) d\lambda =$$

$$= c_9 \int_0^\infty \psi(\lambda) \lambda^n \left(\int_{|y|\geq 2\lambda} |y|^{-n} |f(y)| dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^n \left(\int_{|y|\geq 2\lambda} |y|^{-n}|f(y)|dy\right)^p d\lambda \leq C \int_{\mathbb{R}^n} |f(y)|^p\omega(|y|)dy$$

for $p \in (1,\infty)$ is characterized by the condition $C \leq c\mathcal{B}'$ (see [4], [11]), where

$$\mathcal{B}' \equiv \sup_{\tau>0} \left(\int_0^\tau \psi(t)t^n d\tau\right)\left(\int_{2\tau}^\infty \omega^{1-p'}(t)t^{-np'}dt\right)^{p-1} < \infty.$$

Note that

$$\int_0^\tau \psi(t)t^n dt = n\int_0^\tau \psi(t)dt\int_0^t \lambda^{n-1}d\lambda =$$

$$= n\int_0^\tau \lambda^{n-1}d\lambda\int_\lambda^t \psi(\tau)d\tau \leq n\int_0^\tau \lambda^{n-1}\omega(\lambda)d\lambda = c\int_{|x|<\tau}\omega_1(x)dx.$$

Condition (c') of the theorem guarantees that $\mathcal{B}' \leq n\mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{10}\int_{\mathbb{R}^n} |f(x)|^p\omega(|x|)dx.$$

Combining the estimates of J_1 and J_2, we get (2) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (2). The theorem is proved. □

Theorem 7. *Let $p \in [1,\infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0,\infty)$, $u_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (c). Then inequality (7) is valid.*

Corollary 5. *Let $p \in (1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0,\infty)$, $u_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (c). Then inequality (2) is valid.*

Corollary 6. *Let $p \in [,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $\omega(x) = u(|x|)\phi(x)$, $\omega_1(x) = u_1(|x|)\phi(x)$ be weight functions on $\mathbb{R}^n$, $\phi(x) \in A_p(\mathbb{R}^n)$, $u(t)$ be a weight function on $(0,\infty)$, $u_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair (ω, ω_1) satisfies the conditions (a), (c). Then inequality (7) is valid.*

Theorem 8. *Let $p \in (1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Moreover, $\omega(x')$, $\omega_1(x')$ be weight functions on $\mathbb{R}^m$ and the following three conditions be satisfied:*

(a_1) *there exists a constant $b > 0$ such that*

$$\left(\frac{1}{\omega(x')}\right)\left(\sup_{|x'|/4<|y'|\leq 4|x'|}\omega_1(y')\right) \leq b \quad \textit{for a.e. } x' \in \mathbb{R}^n,$$

$$(b_1)\qquad \mathcal{A}\equiv\sup_{r>0}\left(\int_{|x'|>2r}\omega_1(x')|x'|^{-mp}dx'\right)\left(\int_{|x'|<r}\omega^{1-p'}(x')dx'\right)^{p-1}<\infty,$$

$$(c_1)\qquad \mathcal{B}\equiv\sup_{r>0}\left(\int_{|x'|<r}\omega_1(x')dx'\right)\left(\int_{|x'|>2r}\omega^{1-p'}(x')|x'|^{-mp'}dx'\right)^{p-1}<\infty.$$

Then there exists a constant C, independent of f, such that for all $f\in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}|Tf(x)|^p\omega_1(x')dx\le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x')dx. \tag{8}$$

Moreover, condition (1) can be replaced by the condition

(a_1') *there exists a constant $b>0$ such that*

$$\big(\omega_1(x')\big)\left(\sup_{|x'|/4\le|y'|\le4|x'|}\frac{1}{\omega(y')}\right)\le b\quad \textit{for a.e. } x'\in\mathbb{R}^n.$$

Proof. For $k\in Z$ we define $E_k=\{x\in R^n: 2^k<|x'|\le 2^{k+1}\}$, $E_{k,1}=\{x\in R^n: |x'|<2^{k-1}\}$, $E_{k,2}=\{x\in R^n: 2^{k-1}\le|x'|\le 2^{k+2}\}$, $E_{k,3}=\{x\in R^n: |x'|>2^{k+2}\}$. Given $f\in L_{p,\omega}(\mathbb{R}^n)$, we write

$$|Tf(x)|=\sum_{k\in Z}|Tf(x)|\chi_{E_k}(x)\le\sum_{k\in Z}|Tf_{k,1}(x)|\,\chi_{E_k}(x)+$$

$$+\sum_{k\in Z}|Tf_{k,2}(x)|\,\chi_{E_k}(x)+\sum_{k\in Z}|Tf_{k,3}(x)|\,\chi_{E_k}(x)\equiv T_1f(x)+T_2f(x)+T_3f(x),$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$. We shall estimate $\|T_1f\|_{L_{p,\omega_1}}$. Note that for $x\in E_k$, $y\in E_{k,1}$ we have $|y'|\le 2^{k-1}\le|x'|/2$. Moreover, $|x'-y'|\ge|x'|/2$ and we obtain

$$|T_1f(x)|\le c\sum_{k\in Z}\left(\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{|x-y|^n}dy\right)\chi_{E_k}\le$$

$$\le c\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{|x-y|^n}dy\le c_1\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{(|x'-y'|+|x''-y''|)^n}dy'dy''\le$$

$$\le c_1\int_{\mathbb{R}^{n-m}}\int_{|y'|<|x'|/2}\frac{|f(y)|}{(|x'|+|x''-y''|)^n}dy'dy''$$

for any $x\in E_k$. Using this last inequality we have

$$\int_{\mathbb{R}^n}|T_1f(x)|^p\omega_1(x')dx\le$$

$$\le c_2\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{n-m}}\int_{|y'|<|x'|/2}\frac{|f(y)|}{(|x'|+|x''-y''|)^n}dy'dy''\right)^p\omega_1(x')dx\right\}^{1/p}.$$

For $x=(x',x'')\in\mathbb{R}^n$ let

$$I(x')=\int_{\mathbb{R}^{n-m}}\left(\int_{\mathbb{R}^{n-m}}\int_{|y'|<|x'|/2}\frac{|f(y',y'')|}{(|x'|+|x''-y''|)^n}dy'dy''\right)^p dx''=$$

$$=\int_{\mathbb{R}^{n-m}}\left(\int_{|y'|<|x'|/2}\left(\int_{\mathbb{R}^{n-m}}\frac{|f(y',y'')|}{(|x'|+|x''-y''|)^n}dy''\right)dy'\right)^p dx''.$$

Using the Minkowski and Young inequalities we obtain

$$I(x') \le \left[\int_{|y'|<|x'|/2} \left(\int_{\mathbb{R}^{n-m}} |f(y',y'')|^p dy''\right)^{1/p} \left(\int_{\mathbb{R}^{n-m}} \frac{dx''}{(|x'|+|x''|)^n}\right) dy'\right]^p =$$

$$= \left(\int_{|y'|<|x'|/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p \left(\int_{\mathbb{R}^{n-m}} \frac{dx''}{(|x'|+|x''|)^n}\right)^p =$$

$$= \frac{c_3}{|x'|^{mp}} \left(\int_{|y'|<|x'|/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p \left(\int_{\mathbb{R}^{n-m}} \frac{dx''}{(|x''|+1)^n}\right)^p =$$

$$= \frac{c_4}{|x'|^{mp}} \left(\int_{|y'|<|x'|/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p .$$

Integrating in $\mathbb{R}^m$ we get

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x') dx \le c_4 \int_{\mathbb{R}^m} \omega_1(x') |x'|^{-mp} \left(\int_{|y'|<|x'|/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p dx'.$$

Since $\mathcal{A} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^m} \omega_1(x') |x'|^{-mp} \left(\int_{|y'|<|x'|/5} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p dx' \le$$

$$\le C \int_{\mathbb{R}^m} \|f(x',\cdot)\|_{p,\mathbb{R}^{n-m}} \omega(x') dx'$$

and $C \le c\mathcal{A}$ where c depends only on p. In fact the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [13]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x') dx \le c_5 \int_{\mathbb{R}^n} |f(x)|^p \omega(x') dx.$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1}}$. As is easy to verify, for $x \in E_k$, $y \in E_{k,3}$ we have $|y'| > 2|x'|$ and $|x'-y'| \ge |y'|/2$. For $x \in E_k$ we obtain

$$|T_3 f(x)| \le c_6 \int_{\mathbb{R}^{n-m}} \int_{|y'|>2|x'|} \frac{|f(y)|}{(|y'|+|x''-y''|)^n} dy' dy''.$$

Using this last inequality we have

$$I_1(x') = \int_{\mathbb{R}^{n-m}} \left(\int_{|y'|>2|x'|} \int_{\mathbb{R}^{n-m}} \frac{|f(y)|}{(|y'|+|x''-y''|)^n} dy' dy''\right)^p dx''.$$

Using the Minkowski and Young inequalities we obtain

$$B_1(x') \le \left[\int_{|y'|>2|x'|} \left(\int_{\mathbb{R}^{n-m}} |f(y',y'')|^p dy''\right)^{1/p} \left(\int_{\mathbb{P}^{n-m}} \frac{dy''}{(|y'|+|y''|)^n}\right) dy'\right]^p =$$

$$= c_7 \left(\int_{|y'|>2|x'|} |y'|^{-m} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy'\right)^p \left(\int_{\mathbb{R}^{n-m}} \frac{dy''}{(|y''|+1)^n}\right)^p =$$

$$= c_8 \left(\int_{|y'|>2|x'|} |y'|^{-m} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dy' \right)^p .$$

Using this last inequality we have

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x') dx \le c_8 \int_{\mathbb{R}^m} \left(\int_{|y'|>|7x'|} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} |y'|^{-m} dy' \right)^p \omega_1(x') dx'.$$

Since $\mathcal{B} < \infty$, tee Hardy inequality

$$\int_{\mathbb{R}^m} \omega_1(x') \left(\int_{|y'|>2|x'|} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} |y'|^{-m} dy' \right)^p dx' \le$$

$$\le C \int_{\mathbb{R}^m} \|f(x',\cdot)\|^p_{p,\mathbb{R}^{n-m}} \omega(x') dx' = C \int_{\mathbb{R}^n} |f(x)|^p \omega(x') dx$$

and $C \le c\mathcal{B}$ where c depends only on p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality, (see [1], [13]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x') dx \le c_9 \int_{\mathbb{R}^n} |f(x)|^p \omega(x') dx.$$

Tow, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^n)$ boundedness of T we have

$$\|T_2 f\|_{L_{p,\omega_1}(\mathbb{R}^n)} \le \sum_{k\in Z} \|T f_{k,2}\|_{L_{p,\omega_1}(E_k)} \le c_{10} \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x') \|T f_{k,2}\|_{L_p(E_k)}$$

$$\le c_{11} \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x') \||f_{k,2}\|_{L_p(E_k)} \le c_{12} \sum_{k\in Z} \|f_{k,2}\|_{L_{p,\omega}(E_k)} \le c_{13} \|f\|_{L_{p,\omega}(\mathbb{R}^n)} .$$

Thus

$$\|Tf\|_{L_{p,\omega_1}(\mathbb{R}^n)} \le c\|f\|_{L_{p,\omega}(\mathbb{R}^n)}.$$

We completed the proof of Theorem 8. □

Analogously proved the following weak variant Theorem 8.

Theorem 9. *Let $p \in [1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$, i.e.,*

$$\int_{\{x\in\mathbb{R}^n \,:\, |Tf(x)|>\lambda\}} dx \le \frac{c}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

and satisfying (1)*. Moreover, let ω, ω_1 be weight functions on $\mathbb{R}^m$ and conditions (a_1), (b_1), (c_1) be satisfied.*

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\{x\in\mathbb{R}^n \,:\, |Tf(x)|>\lambda\}} \omega_1(x') dx \le \frac{c}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x') dx. \tag{9}$$

We point out that the condition (1) was first introduced by Soria and Weiss in [15]. The condition (1) is satisfied by many interesting operators in harmonic analysis, such as the Calderon–Zygmund operators, Carleson's maximal operators, Hardy–Littlewood maximal operators, C.Fefferman's singular multipliers, R.Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on; see also [15].

Let K is a Calderon–Zygmund kernel and T be the corresponding integral operator

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^n} K(x-y)f(y)dy.$$

Then T satisfies the conditions of Theorem 2. Thus, we have

Corollary 7. *Let K be a Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $p \in (1,\infty)$, $\omega(x')$,$\omega_1(x')$ be weight functions on $\mathbb{R}^m$ and conditions (a_1), (b_1), (c_1) be satisfied. Then inequality (2) is valid.*

Theorem 10. *Let $p \in (1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Moreover, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and conditions (a_1), (b_1) be satisfied. Then the exists a constant C such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(|x'|)dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(|x'|)dx. \tag{10}$$

Proof. Suppose that $f \in L_p(\mathbb{R}^n, \omega)$ and ω_1 are positive increasing functions on $(0,\infty)$ that satisfy the condition (a_1), (b_1).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)d\lambda,$$

where $\omega_1(0+) = \lim_{t\to 0}\omega_1(t)$ and $\omega_1(t) \ge 0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$.

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_4(|x'|)dx = \omega_1(0+) \int_{\mathbb{R}^n} |Tf(x)|^p dx +$$

$$+ \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_0^{|x'|} \psi(\lambda)d\lambda \right) dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \ne 0$ by the boundedness of T in $L_p(\mathbb{R}^n)$ thanks to (a_1)

$$J_1 \le c\, \omega_1(0+) \int_{\mathbb{R}^n} |f(x)|^p dx \le$$

$$\le c \int_{\mathbb{R}^n} |f(x)|^p \omega_1(|x'|)dx \le c \int_{\mathbb{R}^n} |f(x)|^p \omega(|x'|)dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|w'|>\lambda} |Tf(x)|^p dx \right) d\lambda \le$$

$$\le c \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|>\lambda} |T(f\chi_{\{|x'|>\lambda/2\}})(x)|^p dx \right) d\lambda +$$

$$+c \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|>\lambda} |T(f\chi_{\{|x'|\le\lambda/6\}})(x)|^p dg \right) d\lambda = J_{21} + J_{22}.$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ we obtain

$$J_{21} \le c \int_0^\infty \psi(t) \left(\int_{\mathbb{R}^{n-m}} \int_{|y'|>\lambda/2} |f(y)|^p dy \right) dt =$$

$$= c \int_0^\infty \psi(t) \left(\int_{|y'|>\lambda/2} \|f(y',\cdot)\|^p_{p,\mathbb{R}^{n-m}} dy' \right) dt =$$

$$= c \int_{\mathbb{R}^m} \|f(y',\cdot)\|^p_{p,\mathbb{R}^{n-m}} \left(\int_2^{2|y'|} \psi(\lambda) d\lambda \right) dy' \le$$

$$\le c \int_{\mathbb{R}^m} \|f(y',\cdot)\|^n_{p,\mathbb{R}^{n-m}} \omega_1(2|y'|) dy' \le c_1 \int_{\mathbb{R}^m} |f(y)|^p \omega(|y'|) dy.$$

Let us estimate J_{25}. For $|x'| > \lambda$ and $|y'| \le \lambda/2$ we have $|x'|/2 \le |x'-y'| \le 3|x'|/2$, and so

$$J_{22} \le c \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|>\lambda} \left(\int_{\mathbb{R}^{n-m}} \int_{|y'|\le\lambda/2} \frac{|f(y)|}{|x-y|^n} dy \right)^p dx \right) d\lambda \le$$

$$\le c \int_0^\infty \psi(\lambda) \left(\int_{|x'|>\lambda} \int_{\mathbb{R}^{n-m}} \left(\int_{|y'|\le\lambda/2} \int_{\mathbb{R}^{n-m}} \frac{|f(y)|}{(|x'|+|x''-y''|)^n} dy \right)^p dx \right) d\lambda.$$

For $x = (x', x'') \in \mathbb{R}^n$ let

$$J(x',\lambda) = \int_{\mathbb{R}^{n-m}} \left(\int_{|y'|\le\lambda/2} \int_{\mathbb{R}^{n-m}} \frac{|f(y)|}{(|x'|+|x''-y''|)^n} dy \right)^p dx''.$$

Using the Minkowski and Young inequalities we obtain

$$J(x',\lambda) \le \left[\int_{|y'|\le\lambda/2} \left(\int_{\mathbb{R}^{n-m}} \left(\int_{\mathbb{R}^{n-m}} \frac{|f(y)|}{(|x'|+|x''-y''|)^n} dy'' \right)^p dx'' \right)^{1/p} dy' \right]^p \le$$

$$\le c \left(\int_{|y'|\le\lambda/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{dy''}{(|x'|+|y''|)^n} dy' \right)^p =$$

$$= c|x'|^{-mp} \left(\int_{|y'|\le\lambda/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dx' \right)^p \left(\int_{\mathbb{R}^{n-m}} \frac{dy''}{(1+|y''|)^n} dy'' \right)^p =$$

$$= c|x'|^{-mp} \left(\int_{|y'|\le\lambda/2} \|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}} dx' \right)^p.$$

Integrating in $\mathbb{R}^m$ we get

$$J_{22} \le c\int_0^\infty \psi(\lambda)\left(\int_{|x'|>\lambda}\left(\int_{|y'|\le\lambda/2}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dy'\right)^p|x'|^{-mp}dx\right)d\lambda =$$

$$= c\int_0^\infty \psi(\lambda)\lambda^{-mp+m}\left(\int_{|y'|\le\lambda/4}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dy'\right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{-mp+m}\left(\int_{|y'|\le\lambda/2}|f(y)|dy\right)^p d\lambda \le C\int_{\mathbb{R}^n}|f(y)|^p\omega(|y'|)dy,$$

for $p \in (0,\infty)$ is characterized by the condition $C \le c\mathcal{A}_1$, where

$$\mathcal{A}_1 \equiv \sup_{\tau>0}\left(\int_{2\tau}^\infty \psi(t)t^{-mp+m}d\tau\right)\left(\int_0^\tau \omega^{1-p'}(t)dt\right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-mp+m}d\tau = m(p-1)\int_{2t}^\infty \psi(\tau)d\tau\int_\tau^\infty \lambda^{m-1-mp}d\lambda =$$

$$= m(p-1)\int_{2t}^\infty \lambda^{m-1-mp}d\lambda\int_{2t}^\lambda \psi(\tau)d\tau \le m(p-1)\int_{2t}^\infty \lambda^{m-1-mp}\omega(\lambda)d\lambda.$$

Condition (b_1) of the theorem guarantees that $\mathcal{A}_1 < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c\int_{\mathbb{R}^n}|f(x)|^p\omega(|x'|)dx.$$

Combining the estimates of J_1 and J_1, we get (10) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (10). The Theorem 10 is proved. □

Corollary 8. *Let K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $p \in (1,\infty)$ and ω, ω_1 be weight functions on $(0,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(|x'|), \omega_1(|x'|))$ satisfies the conditions (a_1), (b_1). Then inequality* (10) *is valid.*

Example 1. *Let*

$$\omega(t) = \begin{cases} t^{m(p-1)}\ln^p\frac{1}{t}, & for \quad t \in \left(0,\frac{1}{2}\right) \\ \left(2^{\beta-m(p-1)}\ln^p 2\right)t^\beta, & for \quad t \in \left[\frac{1}{2},\infty\right) \end{cases},$$

$$\omega_1(t) = \begin{cases} t^{m(p-1)}, & for \quad t \in \left(0,\frac{1}{2}\right) \\ 2^{\alpha-m(p-1)}t^\alpha, & for \quad t \in \left[\frac{1}{2},\infty\right) \end{cases},$$

where $0 < \alpha \le \beta < m(p-1)$. Then the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies the condition of Theorem 10.

Analogously proved the following weak variant Theorem 10.

Theorem 11. *Let $p \in [1, \infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let ω, ω_1 be weight functions on $(0, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair $(\omega(|x'|), \omega_1(|x'|))$ satisfies the conditions (a_1), (b_1). Then inequality (9) is valid.*

Corollary 9. *Let $p \in [1, \infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let ω, ω_1 be weight functions on $(0, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair $(\omega(|x'|), \omega_1(|x'|))$ satisfies the conditions (a_1), (b_1). Then inequality (9) is valid.*

Theorem 12. *Let $p \in (1, \infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let ω, ω_1 be weight functions on $(0, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive decreasing function on $(0, \infty)$ and the weighted pair $(\omega(|x'|), \omega_1(|x'|))$ satisfies the conditions (a_1), (c_1). Then inequality (10) is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau) d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of decreasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_3(t)$ for any $t \in (0, \infty)$.

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(|x'|) dx = \omega_1(+\infty) \int_{\mathbb{R}^n} |Tf(x)|^i dx +$$

$$+ \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_{|x'|}^\infty \psi(\tau) d\tau \right) dx = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_2 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of T in $L_p(\mathbb{R}^n)$

$$J_1 \leq c\omega_1(+\infty) \int_{\mathbb{R}^n} |f(x)|^p dx \leq$$

$$\leq c \int_{\mathbb{R}^n} |f(x)|^p \omega_1(|x'|) dx \leq c \int_{\mathbb{R}^n} |f(x)|^p \omega(|x'|) dx.$$

After changing the order of integration in J_2 we have

$$J_5 = \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|<\lambda} |Tf(x)|^p dx \right) d\lambda \leq$$

$$\leq c \int_2^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|<\lambda} |T(f\chi_{\{|x'|<2\lambda\}})(x)|^p dx \right) d\lambda +$$

$$+ c \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-m}} \int_{|x'|<\lambda} |T(f\chi_{\{|x'|\geq 2\lambda\}})(x)|^p dx \right) d\lambda = J_{21} + J_{22}.$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ we obtain

$$J_{21} \le c\int_0^\infty \psi(t)\left(\int_{\mathbb{R}^{n-m}}\int_{|y'|<2\lambda}|f(y)|^p dy\right)dt =$$
$$= c\int_0^\infty \psi(t)\left(\int_{|y'|<2\lambda}\|f(y',\cdot)\|^p_{p,\mathbb{R}^{n-m}}dy'\right)dt =$$
$$= c\int_{\mathbb{R}^m}\|f(y',\cdot)\|^p_{p,\mathbb{R}^{n-m}}\left(\int_{|y'|/2}^\infty \psi(\lambda)d\lambda\right)dy' \le$$
$$\le c\int_{\mathbb{R}^m}\|f(y',\cdot)\|^p_{p,\mathbb{R}^{n-m}}\omega_1(|y'|/3)dy' \le c_1\int_{\mathbb{R}^m}|f(y)|^p\omega(|y'|)dy.$$

Let us estimate J_{22}. For $|x'|<\lambda$ and $|y'|\ge 2\lambda$ we have $|y'|/2 \le |x'-y'| \le 3|y'|/2$, and so

$$J_{22} \le c\int_0^\infty \psi(\lambda)\left(\int_{\mathbb{R}^{n-m}}\int_{|x'|<\lambda}\left(\int_{\mathbb{R}^{n-m}}\int_{|y'|\ge 2\lambda}\frac{|f(y)|}{|x-y|^n}dy\right)^p dx\right)d\lambda \le$$
$$\le c\int_0^\infty \psi(\lambda)\left(\int_{|x'|<\lambda}\int_{\mathbb{R}^{n-m}}\left(\int_{|y'|\ge 2\lambda}\int_{\mathbb{R}^{n-m}}\frac{|f(y)|}{(|x'|+|x''-y''|)^n}dy\right)^p dx\right)d\lambda.$$

For $x=(x',x'')\in\mathbb{R}^n$ let

$$J(x',\lambda) = \int_{\mathbb{R}^{n-m}}\left(\int_{|y'|\ge 2\lambda}\int_{\mathbb{R}^{n-m}}\frac{|f(y)|}{(|x'|+|x''-y''|)^n}dy\right)^p dx''$$

Using the Minkowski and Young inequalities we obtain

$$J(x',\lambda) \le \left[\int_{|y'|\ge 2\lambda}\left(\int_{\mathbb{R}^{n-m}}\left(\int_{\mathbb{R}^{n-m}}\frac{|f(y)|}{(|x'|+|x''-y''|)^n}dy''\right)^p dx''\right)^{1/p}dy'\right]^p \le$$
$$\le c\left(\int_{|y'|\ge 2\lambda}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}\int_{\mathbb{R}^{n-m}}\frac{dy''}{(|x'|+|y''|)^n}\right)^p =$$
$$= c|x'|^{-mp}\left(\int_{|y'|\ge 2\lambda}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dx'\right)^p\left(\int_{\mathbb{R}^{n-m}}\frac{dy''}{(1+|y''|)^n}\right)^p =$$
$$= c|x'|^{-mp}\left(\int_{|y'|\ge 2\lambda}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dx'\right)^p.$$

Integrating in $\mathbb{R}^m$ we get

$$J_{22} \le c\int_0^\infty \psi(\lambda)\left(\int_{|x'|<\lambda}\left(\int_{|y'|\ge 2\lambda}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dy'\right)^p|x'|^{-mp}dx\right)d\lambda =$$
$$= c\int_0^\infty \psi(\lambda)\lambda^{-mp+m}\left(\int_{|y'|\ge 2\lambda}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dy'\right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^m \left(\int_{|y'|\ge 2\lambda} |y'|^{-m}\|f(y',\cdot)\|_{p,\mathbb{R}^{n-m}}dy\right)^p d\lambda \le C\int_{\mathbb{R}^n} |f(y)|^p\omega(|y'|)dy,$$

for $p\in(0,\infty)$ is characterized by the condition $C\le c\mathcal{B}_1$, where

$$\mathcal{B}_1 \equiv \sup_{\tau>0}\left(\int_0^\tau \psi(t)t^m d\tau\right)\left(\int_{2\tau}^\infty \omega^{1-p'}(t)t^{-mp'}dt\right)^{p-1} < \infty.$$

Note that

$$\int_0^\tau \psi(t)t^m dt = m\int_0^\tau \psi(t)dt\int_0^t \lambda^{m-1}d\lambda =$$
$$= m\int_0^\tau \lambda^{m-1}d\lambda\int_\lambda^t \psi(\tau)d\tau \le m\int_0^\tau \lambda^{m-1}\omega(\lambda)d\lambda.$$

Condition (2) of the theorem guarantees that $\mathcal{B}_1<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\le c\int_{\mathbb{R}^n}|f(x)|^p\omega(|x'|)dx.$$

Combining the estimates of J_1 and J_2, we get (10) for $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (10). The Theorem 12 is proved. □

Corollary 10. *Let $p\in[1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let ω, ω_1 be weight functions on $(0,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x'|),\,\omega_1(|x'|))$ satisfies the conditions (a_1), (c_1). Then inequality (10) is valid.*

Analogously proved the following weak variant Theorem 12.

Theorem 13. *Let $p\in[1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying (1). Moreover, let ω, ω_1 be weight functions on $(0,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|),\,\omega_1(|x|))$ satisfies the conditions (a_1), (c_1). Then inequality (9) is valid.*

Corollary 11. *Let $p\in[1,\infty)$, K be a Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let ω, ω_1 be weight functions on $(0,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|),\,\omega_1(|x|))$ satisfies the conditions (a_1), (c_1). Then inequality (9) is valid.*

Example 2. *Let*

$$\omega(t)=\begin{cases} \frac{1}{t^m}\ln^\nu\frac{1}{t}, & for\quad t<d \\ \left(d^{-m-\alpha}\ln^\nu\frac{1}{d}\right)t^\alpha, & for\quad t\ge d \end{cases},$$

$$\omega_1(t) = \begin{cases} \frac{1}{t^m} \ln^\beta \frac{1}{t}, & for \quad t < d \\ \left(d^{-m-\lambda} \ln^\beta \frac{1}{d}\right) t^\lambda, & for \quad t \geq d \end{cases},$$

where $\beta < \nu \leq 0$, $-m < \lambda < \alpha < 0$, $d = e^{\frac{\beta}{m}}$. *Then the weighted pair* $(\omega(|x|), \omega_1(|x|))$ *satisfies the condition of Theorem 12.*

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