Formula for second regularized trace of problem with spectral parameter dependent boundary condition

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Abstract

In the paper we establish formula for the second regularized trace of the problem generated by Sturm – Liouville operator equation and with spectral parameter dependent boundary condition.

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Introduction

Let H be a separable Hilbert space. In the Hilbert space $L_2\left([0,\pi], H\right)$ we consider the following boundary value problem

$$-y''(t) + Ay(t) + q(t)\,y(t) = \lambda y(t), \tag{1}$$

$$y(0) = 0 \tag{2}$$

$$y'(\pi) - \lambda y(\pi) = 0 \tag{3}$$

Here A is a selfadjoint positive definite operator ($A > E$, E is identity operator in H) with a compact inverse, $q(t)$ is a selfadjoint operator-valued function in H for each t. Also let $q(t)$ be weakly measurable with properties:

1. It has fourth order weak derivative on $[0,\pi]$, $q^{(l)}(t) \in \sigma_1(H)$ and $\left\|q^{(l)}(t)\right\|_{\sigma_1(H)} \leq const$ for each $t \in [0,\pi]$, $\left(l = \overline{0,4}\right)$, $Aq^{(l)}(t) \in \sigma_1(H)$ $\left\|Aq^{(l)}(t)\right\| \leq const$ for $l = 0, 2$. Here $\sigma_1(H)$ is a trace class (see [12], p.521, also [9], p.88), class of compact operators in separable Hilbert space, whose singular values form convergent series. It should be noted that in [12] this class is denoted by $\mathcal{B}_1(H)$ while in [9] by $\sigma_1(H)$. We will use the last notation;
2. $q'(0) = q'(\pi) = q(\pi) = 0$;

3. $\int\limits_0^\pi (q(t) f, f)\, dt = 0$ for each $f \in H$.

In direct sum $\mathcal{L}_2 = L_2([0,\pi], H) \oplus H$ let's associate with problem (1)-(3) for $q(t) \equiv 0$ the operator L_0 defined as

$$D(L_0) = \{Y = (y(t), y_1) / y_1 = y(\pi),$$

$$-y''(t) + Ay(t) \in L_2((0,\pi), H),\ \ y(0) = 0\},$$

$$L_0 Y = (-y''(t) + Ay(t),\ \ y'(\pi)).$$

Let's denote by L the perturbed operator: $L = L_0 + Q$, where

$$Q(y(t), y(\pi)) = (q(t) y(t), 0).$$

It is known that [16] operators L_0 and L have a discrete spectrum. Denote their eigenvalues by $\mu_1 \le \mu_2 \le ...$ and $\lambda_1 \le \lambda_2 \le ...$, respectively.

The main goal of the paper is to establish a formula for second regularized trace of operator L. A formula for the first regularized trace of operator L is obtained in [2].

The formula of regularized trace of Sturm – Liouville operator was first obtained by I.M.Gelfand and B.M.Levitan (see [8]).

After this work, numerous investigations on calculation of regularized trace of concrete operators, as well as differential operator equations and discrete abstract operators appeared (see, for example, [2]-[8], [11], [13]-[15], [17]-[19]). One can find additional references on the subject in [19].

Individual approach to concrete problems gives sometimes stronger results in comparison with general theorems. Results for operators generated by differential operator equations have applications to concrete problems of mathematical physics.

It should be noted that one of the applications of trace formulas is approximate calculations of first eingenvalues of differential operators ([4], [5]) and inverse problems ([14]).

1. Preliminaries

Let's denote the eigenvectors and eigenvalues of operator A by $\varphi_1, \varphi_2, ...$ and $\gamma_1 \le \gamma_2 \le ...$ respectively. It is known that (see [16]) if $\gamma_i \sim a \cdot i^\alpha, a > 0$, $\alpha > 2$ then

$$\mu_k \sim \lambda_k \sim k^{\frac{2\alpha}{2+\alpha}}. \tag{1.1}$$

Let R_λ^0 be resolvent of operator L_0^2. In view of asymptotics for μ_k, it follows that R_λ^0 is from $\sigma_1(H)$. In [18] the following theorem was proved

Theorem 1. *Let $D(A_0) \subset D(B)$, where A_0 is a selfadjoint positive discrete operator in separable Hilbert space H, such that $A_0^{-1} \in \sigma_1(H)$ and let B be a perturbation operator. Assume that there exist a number $\delta \in [0,1)$ such that $BA_0^{-\delta}$ is continuable to bounded operator and some number $\omega \in [0,1)$, $\omega + \delta < 1$, such that $A_0^{-(1-\delta-\omega)}$ is a trace class operator. Then there exist subsequence of natural numbers $\{n_m\}_{m=1}^{\infty}$ and sequence of closed contours $\Gamma_m \in \mathbb{C}$ such that for $N \geq \frac{\delta}{\omega}$*

$$\lim_{m\to\infty}\left(\sum_{j=1}^{n_m}(\mu_j-\lambda_j)+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}T_r\left(BR_0(\lambda)\right)^k d\lambda\right)=0$$

(here $\{\mu_n\}$ and $\{\lambda_n\}$ are eigenvalues of A_0+B and A_0, respectively, arranged in ascending order of their real parts, $R_0(\lambda)$ is a resolvent of A_0).

In particular, for $\omega \geq \delta$ it holds

$$\lim_{m\to\infty}\sum_{j=1}^{n_m}(\mu_j-\lambda_j-(B\varphi_j,\varphi_j))=0,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is a basis formed by eigenvectors of A_0.

The conditions of this theorem are satisfied for L_0^2 and L^2. Really, if we take $A_0 = L_0^2$, $B = L_0Q + QL_0 + Q^2$ ($L^2 = A_0 + B$) and $\delta = \frac{1}{2}$, provided $L_0QL_0^{-1}$ is bounded, BA_0^{-1} is also bounded and for $\omega \in [0,1)$, $\omega < \frac{1}{2} - \frac{2+\alpha}{4\alpha}$,

$$A_0^{-(1-\delta-\omega)} = L_0^{-2(1-\delta-\omega)}$$

is an operator of the trace class because of asymptotics (1.1). Thus by statement of Theorem 1 for $N > \dfrac{1}{2\omega}$

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right)=0. \qquad (1.2)$$

2. Regularized trace

Let's call

$$\lim_{m\to\infty}\left\{\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\frac{1}{\pi}\int_0^{\pi} trq^2(t)dt\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right\} \quad (2.1)$$

a second regularized trace of L and denote it by $\sum\limits_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)$. Further, we will show that it has finite value which doesn't depend on choice of $\{n_m\}$.

By virtue of [18, lemma 3] for great m the number of eigenvalues of L_0^2 and L^2 inside the contour Γ_m is the same and equals to n_m.

In view of (1.2)

$$\lim_{m\to\infty}\left\{\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\frac{1}{\pi}\int_0^{\pi} trq^2(t)dt\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}\left(tr\left[(L_0Q+QL_0+Q^2)R_0(\lambda)\right]^k\right)d\lambda\right\}=$$

$$=\lim_{m\to\infty}\left(-\frac{1}{2\pi\, i}\int_{\Gamma_m} tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]d\lambda-\right.$$

$$\left.-\sum_{n=1}^{n_m}\frac{1}{\pi}\int_0^{\pi} trq^2(t)\,dt\right). \quad (2.2)$$

Denote the eigenvectors of L_0 by $\psi_1,\psi_2,\cdots$. By our assumption operator $L_0QL_0^{-1}$ is bounded, so $[L_0Q+QL_0+Q^2]\,R_\lambda^0$ is an operator of trace class and since eigenvectors of L_0 form a basis in $\mathcal{L}_2$, we can change the first term on the right – hand side of (2.2) in the following way:

$$\frac{1}{2\pi\, i}\int_{\Gamma_m} tr\left[\left(QL_0+L_0Q+Q^2\right)R_0(\lambda)\right]d\lambda=$$

$$= \frac{1}{2\pi\, i} \int\limits_{\Gamma_m} \sum_{n=1}^{\infty} \left(\left(QL_0 + L_0 Q + Q^2 \right) R_0 \left(\lambda \right) \psi_n , \psi_n \right)_{\mathcal{L}_2} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int\limits_{\Gamma_m} \frac{1}{\mu_n^2 - \lambda} \left(\left(QL_0 + L_0 Q + Q^2 \right) R_0 \left(\lambda \right) \psi_n , \psi_n \right)_{\mathcal{L}_2} d\lambda =$$

$$= - \sum_{n=1}^{n_m} \left(\left[QL_0 + L_0 Q + Q^2 \right] \psi_n , \psi_n \right)_{\mathcal{L}_2} . \tag{2.3}$$

Note that the eigenvectors $\{\psi_n\}_{n=1}^{\infty}$ are of the form (see [2])

$$\sqrt{\frac{4x_{j,k}}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi}} \left\{ \sin\left(x_{j,k} t\right)\varphi_j , \sin\left(x_{j,k}\pi\right)\varphi_j \right\} , \tag{2.4}$$

$$\left\{ \begin{array}{l} k = \overline{1,\infty},\ j = \overline{1,\infty} \\ k = 0, \ \ j = \overline{N,\infty}, \end{array} \right.$$

where $x_{j,k}$ are the roots (see [16]) of the equation

$$ctg\, x\pi = \frac{\gamma_j + x^2}{x}, \quad x = \sqrt{\lambda - \gamma_j}. \tag{2.5}$$

It is known that eigenvalues of L_0 form two sequences: $\mu_{j,0} \sim \sqrt{\gamma_j}$, as $j \to \infty$, which correspond to imaginary roots of (2.5) and $\mu_{j,k} = \gamma_j + x_{j,k}^2 = \gamma_j + \eta_k, \eta_k \sim k^2$ which correspond to real roots of (2.5). To calculate regularized trace, the following lemma will be required.

Lemma 2.1. *If properties 1,2 hold, and $\gamma_j \sim aj^{\alpha}, a > 0, \alpha > 2$, then the following series is absolutely convergent*

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \left(\gamma_j + x_{j,k}^2\right) \frac{2x_{j,k} \int\limits_0^{\pi} \cos 2x_{j,k} t f_j(t) dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi} \right| +$$

$$+ \sum_{j=N}^{\infty} \left| \frac{\left(\gamma_j + x_{j,0}^2\right) 2x_{j,0} \int\limits_0^{\pi} \cos 2x_{j,0} t f_j(t) dt}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi} \right| +$$

$$+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\left|\frac{4x_{j,k}\int\limits_0^{\pi}\sin^2 x_{j,k}tg_j(t)dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)dt\right|\right)+$$

$$+\sum_{j=N}^{\infty}\left(\left|\int\limits_0^{\pi}\frac{4x_{j,0}\int\limits_0^{\pi}\sin^2 x_{j,0}tg_j(t)dt}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^2 x_{j,0}\pi}-\right.\right.$$

$$\left.\left.-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)dt\right|\right)<\infty, \tag{2.6}$$

where $f_j(t)=(q(t)\varphi_j,\varphi_j)\,,g_j(t)=(q^2(t)\varphi_j,\varphi_j)\,.$

Proof. Let's denote the sums on the left of (2.6) by s_1, s_2, s_3, s_4 according to their order. By virtue of property 2, integrating by parts at first twice, then four times, we have

$$\int\limits_0^{\pi}\cos 2x_{j,k}tf_j\,(t)\,dt=-\frac{1}{(2x_{j,k})^2}\int\limits_0^{\pi}\cos 2x_{j,k}t\,f_j''\,(t)\,dt \tag{2.7}$$

$$\int\limits_0^{\pi}\cos 2x_{j,k}tf_j\,(t)\,dt=$$

$$=-\frac{1}{(2x_{j,k})^3}f_j''\,(\pi)\sin 2x_{j,k}\pi-\frac{1}{(2x_{j,k})^4}\cos 2x_{j,k}tf_j'''\,(t)\Big|_0^{\pi}+$$

$$+\frac{1}{(2x_{j,k})^4}\int\limits_0^{\pi}\cos 2x_{j,k}t\,f_j^{(IV)}\,(t)\,dt. \tag{2.8}$$

In virtue of estimate

$$\frac{2x_{j,0}}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^2 x_{j,0}\pi}=\frac{1}{\pi}+O\left(\frac{1}{x_{j,0}}\right) \tag{2.9}$$

using property 1 and relation (2.7) we have

$$\sum_{j=N}^{\infty}\left|\frac{2x_{j,0}\gamma_j\int\limits_0^{\pi}\cos 2x_{j,0}tf_j\left(t\right)dt}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^2 x_{j,0}\pi}\right|\leq$$

$$\leq\sum_{j=N}^{\infty}\gamma_j\left(\frac{1}{\pi}+O\left(\frac{1}{x_{j,0}}\right)\right)\int\limits_0^{\pi}\left|f_j\left(t\right)\right|dt<\infty$$

$$\sum_{j=N}^{\infty}\left|\frac{2x_{j,0}^3\int\limits_0^{\pi}\cos 2x_{j,0}tf_j\left(t\right)dt}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^2 x_{j,0}\pi}\right|\leq$$

$$\leq\sum_{j=N}^{\infty}\left(\frac{1}{2\pi}+O\left(\frac{1}{x_{j,0}}\right)\right)\int\limits_0^{\pi}\left|f_j''\left(t\right)\right|dt.$$

So, we get that series denoted by s_2 is absolutely convergent.

Then by virtue of (2.8), asymptotics $x_{j,k}\sim k+\dfrac{k}{\gamma_j+k^2}$, property $\left\|Aq''\left(t\right)\right\|_1\leq$ *const* (norm in $\sigma_1(H)$ we denote simply by $\|\cdot\|_1$) and (2.7) the following estimate holds

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2\gamma_j x_{j,k}\int\limits_0^{\pi}\cos 2x_{j,k}tf_j\left(t\right)dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\gamma_j\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)O\left(\frac{1}{k^2}\right)\int\limits_0^{\pi}\left|f_j''\left(t\right)\right|dt=$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}O\left(\frac{1}{k^2}\right)\int\limits_0^{\pi}\left|\left(Aq''\left(t\right)\varphi_j,\varphi_j\right)\right|dt<const \qquad (2.10)$$

Since $\left\|q^{(l)}(t)\right\|_1 \le const\,(l=2,4)$, again by using asymptotics for $x_{j,k}$ and (2.8) we obtain

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}^3\int\limits_0^\pi \cos 2x_{j,k}t f_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\right| = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\times$$

$$\times\left[\frac{1}{x_{j,k}}\left|f_j''(\pi)\sin 2x_{j,k}\pi\right| + \frac{1}{(2x_{j,k})^2}\left(\left|f_j''(\pi)\right|+\left|f_j''(0)\right|\right)+\right.$$

$$\left.+\frac{1}{(2x_{j,k})^2}\int\limits_0^\pi \left|f_j^{(IV)}(t)\right|dt\right] < \infty. \tag{2.11}$$

Here it was also used that $\sin(2x_{j,k}\pi)\sim \dfrac{1}{k}$.

From (2.10) and (2.11) it follows that series denoted by s_1 is also convergent.

Then

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{4x_{j,k}\int\limits_0^\pi \sin^2(x_{j,k}t)g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right| =$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\,g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right| =$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\int\limits_0^\pi (1-cos2x_{j,k}t)g_j(t)dt-\frac{1}{\pi}\int\limits_0^\pi g_j(t)dt\right| =$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\int\limits_0^\pi \cos 2x_{j,k}t g_j(t)\,dt+O\left(\frac{1}{k^2}\right)\int\limits_0^\pi g_j(t)\,dt\right|.$$

The last equality in virtue of (2.7) and properties $g_j''(t)\in\sigma_1(H)$, $g_j(t)\in\sigma_1(H)$ gives that series denoted by s_3 converges. Similarly it can be shown that s_4 also converges and this completes the proof of the lemma.

Now let's calculate the value of series called the second regularized trace. For that we prove the following theorem.

Assume that

$$\int\limits_{\pi-\delta}^{\pi} \frac{g_j(t)}{\pi - t} dt < \infty \tag{2.12}$$

for small $\delta > 0$.

Theorem 2.1 *Let $q(t)$ be an operator-function with properties 1-3, $L_0^{-1} Q L_0$ be bounded operator in $\mathbb{L}_2$, and $\gamma_j \sim a \cdot j^{\alpha}$, $a > 0$, $\alpha > 2$, then provided that (2.12) holds*

$$\sum_{n=0}^{\infty} \left(\lambda_n^{(2)} - \mu_n^{(2)}\right) = -\frac{trq^2(0)}{4} -$$

$$-\frac{tr Aq(0) + tr Aq(\pi)}{2} + \frac{trq''(0) + trq''(\pi)}{8}. \tag{2.13}$$

Proof. It follows from lemma 2.1 and relations (2.2) and (2.3) that

$$\lim_{m\to\infty} \left\{ \sum_{n=1}^{n_m} \left(\lambda_n^2 - \mu_n^2 - \frac{1}{\pi} \int\limits_0^{\pi} trq^2(t)\, dt \right) + \right.$$

$$\left. + \frac{1}{2\pi i} \int\limits_{\Gamma_m} \sum_{k=2}^{N} \frac{(-1)^{k-1}}{k} tr \left[(L_0 Q + Q L_0 + Q^2) R_0(\lambda) \right]^k d\lambda \right\} =$$

$$= \sum_{j=1}^{N-1} \sum_{k=1}^{\infty} 2\left(\gamma_j + x_{j,k}^2\right) \frac{4x_{j,k} \int_0^{\pi} \frac{1-\cos 2x_{j,k}t}{2} f_j(t) dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k} \sin^2 x_{j,k}\pi} +$$

$$+ \sum_{j=N}^{\infty} \sum_{k=0}^{\infty} 2\left(\gamma_j + x_{j,k}^2\right) \frac{4x_{j,k} \int_0^{\pi} \frac{1-\cos 2x_{j,k}t}{2} f_j(t) dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k} \sin^2 x_{j,k}\pi} +$$

$$+ \sum_{j=1}^{N-1} \sum_{k=1}^{\infty} \left[\frac{2x_{j,k} \int_0^{\pi} (1 - \cos 2x_{j,k}t)\, g_j(t)\, dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k} \sin^2 x_{j,k}\pi} - \frac{1}{\pi} \int\limits_0^{\pi} g_j(t) dt \right] +$$

$$+\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^{\pi}(1-\cos 2x_{j,k}t)\,g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)dt\right]=$$

$$=-\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^{\pi}\cos 2x_{j,k}tf_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-$$

$$-\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^{\pi}\cos 2x_{j,k}tf_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}+$$

$$+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^{\pi}(1-\cos 2x_{j,k}t)\,g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)\,dt\right]+$$

$$+\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^{\pi}(1-\cos 2x_{j,k}t)\,g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)\,dt\right]. \quad (2.14)$$

At first derive a formula for the fourth term on the right of (2.14). For that consider

$$\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^{\pi}g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)\,dt\right].$$

Let's calculate the value of the inner series for each fixed j

$$\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\right]=$$

$$=\lim_{N\to\infty}\sum_{k=0}^{N-1}\left[\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\right] \quad (2.15)$$

Denote the partial sum of above series by T_N and investigate its behavior as $N\to\infty$. Let's express the k-th term of the sum T_N as a residue at a pole

$x_{j,k}$ of some function of complex variable z for which $x_{j,0},..,x_{j,N}$ are poles. Thus, consider the following complex-valued function

$$g\left(z\right)=\frac{-z}{\left(zctgz\pi-z^{2}-\gamma_{j}\right)\sin^{2}z\pi} \tag{2.16}$$

for which as it is easy to see that $x_{j,k}$ and k are simple poles. The residue at the point $x_{j,k}$ is

$$\underset{z=x_{j,k}}{res}\; g\left(z\right)=\frac{-x_{j,k}}{\left(ctgx_{j,k}\pi-\dfrac{\pi x_{j,k}}{\sin^{2}x_{j,k}\pi}-2x_{j,k}\right)\sin^{2}x_{j,k}\pi}=$$

$$=\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^{2}x_{j,k}\pi},$$

at the point k is

$$\underset{z=k}{res}\, g\left(z\right)=\frac{-k}{\left(k\cos k\pi-k^{2}\sin k\pi-\gamma_{j}\sin k\pi\right)\pi\cos\pi k}=-\frac{1}{\pi}.$$

Now take a rectangular contour of integration with vertices at points $\pm iB$, $A_N\pm iB$, which has cut at $ix_{j,0}$ and will pass it by on the left, and the points $-ix_{j,0}$ and 0 on the right. Take also $B>x_{j,0}$. Then B will go to infinity and $A_N=N+\frac{1}{2}$. For this choice of A_N we have $x_{j,N-1}<A_N<x_{j,N}$, and the number of points $x_{j,k}$ inside of contour of integration equals N $\left(k=\overline{0,N-1}\right)$.

One could easily show that inside this contour the function $zctgz\pi-z^2-\gamma_j$ has exactly N roots, so $x_{j,N-1}<A_N<x_{j,N}$.

Function (2.16) is an odd function of z, that's why the integral along the part of the contour on imaginary axis as well along semicircles centered at $\pm x_{j,0}$ vanishes.

If $z=u+iv$ then for large v and $u\geq 0$, the order of $g\left(z\right)$ is $O\left(e^{-2\pi|v|}\right)$, and for chosen A_N the integrals along upper and lower sides of contour go to zero as $B\to\infty$

So, we come to the following equality

$$T_N=\frac{1}{2\pi i}\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB}\frac{-zdz}{\left(zctgz\pi-z^{2}-\gamma_{j}\right)\sin^{2}z\pi}+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{-zdz}{(zctgz\pi - z^2-\gamma_j)\sin^2 z\pi}\ , \tag{2.17}$$

where in the second integral $z = re^{i\varphi}$.

As $N \to \infty$, the first term on the right of (2.17) is equivalent to

$$\frac{1}{\pi i}\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{dz}{2z\sin^2 z\pi-\sin 2z\pi}=$$

$$=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{d\upsilon}{2\left(A_N+i\upsilon\right)\left(1+ch2\upsilon\pi\right)-\dfrac{sh2\upsilon\pi}{i}}\ , \tag{2.18}$$

whose absolute value is less then

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{d\upsilon}{2\left|A_N+i\upsilon\right|\left(1+ch2\upsilon\pi\right)-\left|sh2\upsilon\pi\right|}=$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{A_N^2+\upsilon^2}\left(1+ch2\upsilon\pi\right)-\dfrac{sh2\upsilon\pi}{2}}d\upsilon<$$

$$<\frac{1}{2A_N\pi}\int\limits_{-\infty}^{\infty}\frac{d\upsilon}{\left(1+ch2\upsilon\pi\right)-\dfrac{sh2\upsilon\pi}{2\sqrt{A_N^2+\upsilon^2}}}<$$

$$<\frac{1}{2A_N\pi}\int\limits_{-\infty}^{\infty}\frac{d\upsilon}{1+ch2\upsilon\pi-\dfrac{1+ch2\upsilon\pi}{2}}=\frac{const}{A_N}. \tag{2.19}$$

Therefore,

$$\int\limits_{0}^{\pi}T_N g_j(t)dt=-\frac{1}{2\pi i}\int\limits_{0}^{\pi}g_j\left(t\right)\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{zdzdt}{(zctgz\pi-z^2-\gamma_j)\sin^2 z\pi}-$$

$$-\frac{1}{2\pi i}\lim_{r\to 0}\int_0^\pi g_j(t)\,dt \int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{zdz}{(zctgz\pi - z^2-\gamma_j)\sin^2 z\pi}. \tag{2.20}$$

But as $r\to 0$

$$-\frac{1}{2\pi i}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{zdz}{(zctgz\pi - z^2-\gamma_j)\sin^2 z\pi}\sim$$

$$\sim -\frac{1}{2\pi i}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{zdz}{z\sin z\pi-\gamma_j\sin^2 z\pi}=$$

$$=-\frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ir^2e^{2i\varphi}d\varphi}{r^2e^{2i\varphi}\pi-\gamma_j\pi^2r^2e^{2i\varphi}}=-\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{\pi-\gamma_j\pi^2}=-\frac{1}{2\pi}\frac{1}{1-\gamma_j\pi}. \tag{2.21}$$

So, using (2.17), (2.18), (2.19) and (2.21) in (2.20) we have

$$\lim_{N\to\infty}\int_0^\pi T_N g_j(t)\,dt = -\frac{1}{2\pi(1-\gamma_j\pi)}\int_0^\pi g_j(t)\,dt. \tag{2.22}$$

Now let's derive calculations for

$$S_N(t) = -\sum_{k=0}^{N-1}\frac{2x_{jk}\int\limits_0^\pi \cos 2x_{jk}tg_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}.$$

Consider the complex valued function

$$G(z)=\frac{z\cos 2zt}{(zctgz\pi-z^2-\gamma_j)\sin^2 z\pi}$$

whose residues at the poles k and $x_{j,k}$ equal $\dfrac{2x_{j,k}\cos 2x_{j,k}t}{\sin 2x_{j,k}\pi-2x_{j,k}\pi-4x_{j,k}\sin^2 2x_{j,k}\pi}$ and $\dfrac{\cos 2kt}{\pi}$, respectively. Again take as a contour of integration the above considered contour. One could show that as $N \to \infty$

$$\frac{1}{2\pi i} \int\limits_{A_N - i\infty}^{A_N + i\infty} G(z)\, dz \sim \frac{const}{A_N \cos \frac{t}{2}}. \tag{2.23}$$

Thus, if $g_j(t)$ has the property (2.12), then

$$\lim_{N \to \infty} \int\limits_0^{\pi} g_j(t) \int\limits_{A_N - i\infty}^{A_N + i\infty} G(z)\, dz dt = 0. \tag{2.24}$$

In virtue of (2.24)

$$\lim_{N \to \infty} \int\limits_0^{\pi} S_N(t)\, g_j(t)\, dt = - \lim_{N \to \infty} \int\limits_0^{\pi} M_N(t)\, g_j(t)\, dt +$$

$$+ \frac{1}{2\pi i} \lim_{r \to 0} \int\limits_0^{\pi} g_j(t) \int\limits_{\substack{|z| = r \\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}}} \frac{z \cos 2zt}{(zctgz\pi - z^2 - \gamma_j) \sin^2 z\pi} dt, \tag{2.25}$$

where

$$M_N(t) = \sum_{k=1}^{N} \frac{\cos 2kt}{\pi}.$$

Since

$$\lim_{N \to \infty} \int\limits_0^{\pi} M_N(t)\, g_j(t)\, dt = \frac{1}{\pi} \sum_{k=0}^{\infty} \int\limits_0^{\pi} g_j(t) \cos 2kt dt = \frac{g_j(\pi) + g_j(0)}{4},$$

and the second term in (2.25) as $r \to 0$ goes to $\frac{1}{2\pi(1 - \gamma_j \pi)} \int\limits_0^{\pi} g_j(t)\, dt$, then

$$\lim_{N \to \infty} \int\limits_0^{\pi} S_N(t)\, g_j(t)\, dt = - \frac{g_j(\pi) + g_j(0)}{4} + \frac{1}{2\pi(1 - \gamma_j \pi)} \int\limits_0^{\pi} g_j(t)\, dt. \tag{2.26}$$

Combining (2.22) and (2.26), we get

$$\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2x_{jk}\int\limits_0^\pi\left(1-\cos 2x_{jk}t\right)g_j\left(t\right)dt}{2x_{jk}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j\left(t\right)dt\right)=$$

$$=\sum_{j=N}^{\infty}-\frac{g_j\left(\pi\right)+g_j\left(0\right)}{4}+\int\limits_0^\pi\left(\frac{g_j\left(t\right)}{2\pi\left(1-\gamma_j\pi\right)}-\frac{g_j\left(t\right)}{2\pi\left(1-\gamma_j\pi\right)}\right)dt=$$

$$=-\sum_{j=N}^{\infty}\frac{g_j\left(\pi\right)+g_j\left(0\right)}{4}=-\sum_{j=N}^{\infty}\frac{g_j(0)}{4}. \tag{2.27}$$

Here the condition

$$g_j\left(\pi\right)=\left(q^2\left(\pi\right)\varphi_j,\varphi_j\right)=\left(q\left(\pi\right)\varphi_j,q\left(\pi\right)\varphi_j\right)=0$$

is used.

By similar computations (this time contour of integration by–passes only the origin along small semicircle, since this time the chosen complex function has no imaginary roots), we will have

$$\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left(\frac{2x_{jk}\int\limits_0^\pi\left(1-\cos 2x_{jk}t\right)g_j\left(t\right)dt}{2x_{jk}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\right.$$

$$\left.-\frac{1}{\pi}\int\limits_0^\pi g_j\left(t\right)dt\right)=-\sum_{j=1}^{N-1}\frac{g_j\left(0\right)}{4} \tag{2.28}$$

From (2.27) and (2.28) the sum of values of two last series in (2.14) gives

$$-\sum_{j=1}^{N-1}\frac{g_j\left(0\right)}{4}-\sum_{j=N}^{\infty}\frac{g_j\left(0\right)}{4}=-\frac{trq^2\left(0\right)}{4}.$$

By the method used above, we may derive all calculations also for the first two series in (2.14) and come finally to formula (2.13).

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