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IMPULSIVE q-STURM-LIOUVILLE PROBLEMS

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In this study, impulsive q-Sturm–Liouville problems are considered. First, symmetry is obtained with the help of boundary conditions. Then, the existence and uniqueness problem for such equations is discussed. Finally, eigenfunction expansion was obtained with the help of characteristic determinant and Green's function.

1. INTRODUCTION

The Sturm–Liouville problems have a long history. Such problems have been studied for a long time. Sturm–Liouville problems arise, especially if it is desired to solve partial differential equations modeling various problems encountered in different fields of science with the Fourier method. For more detailed information on Sturm–Liouville problems, see ([**15**]). On the other hand, we encounter impulsive Sturm–Liouville problems in geophysics, electromagnetics, elasticity, and other fields of engineering and physics. For problems of this type see ([**4, 5, 16, 6**]).

Quantum calculus has recently started to attract a lot of attention. The fact that some functions that cannot be differentiated in the classical sense can be differentiated in the quantum sense makes this subject interesting. Various problems involving differentiable functions in the quantum sense can be encountered in different fields of mathematics ([**8**]). In 2005, Annaby and Mansour applied quantum calculus to classical Sturm–Liouville problems and investigated q-Sturm–Liouville problems ([**2**]). Later on, q-Sturm–Liouville problems were studied by some authors by putting impulsive boundary conditions. In [**7**], Çetinkaya studied discontinuous

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q-Sturm–Liouville problems with eigenparameter-dependent boundary conditions. In [**11, 12, 13**], Karahan and Mamedov investigated a q-Sturm–Liouville problem with discontinuity conditions. In [**14**], the author studied the singular q-Sturm–Liouville problem with impulsive conditions.

In this paper, we study impulsive q-Sturm–Liouville problems. Firstly, the fundamental spectral properties of these problems are obtained. Later, the existence and uniqueness problem for such equations is discussed. Finally, eigenfunction expansion is obtained with the help of characteristic determinant and Green's function.

2. PRELIMINARIES

In this section, the basic concepts of q-calculus that will be used in the article will be given. For more detailed information, the following sources can be examined, [**10, 3, 8, 9**].

Let $q \in (0,1)$ and let $A \subset \mathbb{R}$ be a q-geometric set, i.e., if $q\zeta \in A$ for all $\zeta \in A$. We begin by defining the *operator* $\mathcal{D}_q$ by

$$\mathcal{D}_q f(\zeta) = \begin{cases} \frac{f(q\zeta)-f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0 \\ \lim_{n\to\infty} \frac{f(q^n\xi)-f(0)}{q^n\xi}, & \zeta = 0, \end{cases}$$

where $\zeta, \xi \in A$. When it is required, q will be replaced by q^{-1}. The following facts, which will be frequently used, can be verified directly from the definition:

$$\mathcal{D}_{q^{-1}} f(\zeta) = (\mathcal{D}_q f)(q^{-1}\zeta),\ (\mathcal{D}_q^2 f)(q^{-1}\zeta) = q\mathcal{D}_q[\mathcal{D}_q f(q^{-1}\zeta)] = \mathcal{D}_{q^{-1}}\mathcal{D}_q f(\zeta).$$

Related to this operator there exists a non-symmetric formula for the q-differentation of a product

$$\mathcal{D}_q[f(\zeta)g(\zeta)] = g(\zeta)\mathcal{D}_q f(\zeta) + f(q\zeta)\mathcal{D}_q g(\zeta).$$

We define the *Jackson q-integration* by

$$\int_0^\zeta f(\gamma)\, d_q\gamma = \zeta(1-q)\sum_{n=0}^{\infty} q^n f(q^n\zeta) \ \ (\zeta \in A),$$

provided that the series converges, and

$$\int_a^b f(\gamma)\, d_q\gamma = \int_0^b f(\gamma)\, d_q\gamma - \int_0^a f(\gamma)\, d_q\gamma,$$

where $a, b \in A$. Through the remainder of the paper, we deal only with functions q-regular at zero, i.e, functions satisfying

$$\lim_{n\to\infty} f(\zeta q^n) = f(0),$$

for every $\zeta \in A$.

Let

$$L_q^2(0,a) = \left\{ f : [0,a] \to \mathbb{C} : \sqrt{\int_0^a |f(\zeta)|^2 \, d_q\zeta} < \infty \right\},$$

$L_q^2(0,a)$ is a Hilbert space endowed with the inner product

$$(f,g) := \int_0^a f(\zeta)\, \overline{g(\zeta)} d_q\zeta, \ \|f\| := \sqrt{\int_0^a |f(\zeta)|^2 \, d_q\zeta}.$$

The q-trigonometric functions are given by the formulas

$$\cos(z;q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2} (z(1-q))^{2n}}{(q;q)_{2n}},$$

$$\sin(z;q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)} (z(1-q))^{2n+1}}{(q;q)_{2n+1}},$$

where

$$(a;q)_0 = 1, \ (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right)$$

(see [**2**]).

The q-Wronskian of the functions y and z is defined by the formula

$$W_q(y,z) := yD_q z - zD_q y.$$

3. STATEMENT OF THE PROBLEM

Let us consider the following q-Sturm–Liouville equation

$$\Upsilon(y) := \left[-\frac{1}{q} \mathcal{D}_{q^{-1}} \mathcal{D}_q + v(\zeta) \right] y(\zeta) = \lambda y(\zeta), \ \zeta \in [0,d) \cup (d,a], \tag{1}$$

subject to the following conditions

$$y(0) + k_1 \mathcal{D}_{q^{-1}} y(0) = 0, \tag{2}$$

$$y(d-) - k_2 y(d+) = 0, \tag{3}$$

$$\mathcal{D}_{q^{-1}} y(d-) - k_3 \mathcal{D}_{q^{-1}} y(d+) = 0, \tag{4}$$

$$y(a) + k_4 \mathcal{D}_{q^{-1}} y(a) = 0, \tag{5}$$

where k_1, k_2, k_3, k_4 are real numbers and λ is a complex parameter.

Our basic assumption throughout the paper is the following:

(K1) Let $q \in (0,1)$, $k_2 k_3 = \alpha > 0$ and v is a real-valued function that is continuous on $[0,d) \cup (d, q^{-1}a]$ and has finite limits $v(d\pm)$.

Let us introduce the following space:

$H = L_q^2(0,d) \dotplus L_q^2(d,a)$ is a Hilbert space endowed with the following inner product

$$\langle f, g\rangle_H := \int_0^d f^{(1)}\overline{g^{(1)}} d_q\zeta + \alpha \int_d^a f^{(2)}\overline{g^{(2)}} d_q\zeta,$$

where

$$f(\zeta) = \begin{cases} f^{(1)}(\zeta), & \zeta \in [0,d) \\ f^{(2)}(\zeta), & \zeta \in (d,a], \end{cases} \quad g(\zeta) = \begin{cases} g^{(1)}(\zeta), & \zeta \in [0,d) \\ g^{(2)}(\zeta), & \zeta \in (d,a]. \end{cases}$$

Consider the following sets

$$D_{\max} = \left\{ y \in H : \begin{array}{c} \text{one-sided limits } y(d\pm) \text{ and } \mathcal{D}_{q^{-1}} y(d\pm) \\ \text{exist and finite, } y(d-) - k_2 y(d+) = 0, \\ \mathcal{D}_{q^{-1}} y(d-) - k_3 \mathcal{D}_{q^{-1}} y(d+) = 0, \text{ and } \Upsilon y \in H \end{array} \right\},$$

$$D_{\min} = \left\{ y \in D_{\max} : y(0) = \mathcal{D}_{q^{-1}} y(0) = y(a) = \mathcal{D}_{q^{-1}} y(a) = 0 \right\}.$$

Then the *maximal* operator $\mathcal{L}_{\max}$ on $D_{\max}$ is defined by

$$\mathcal{L}_{\max} y = \Upsilon(y).$$

If we restrict the operator $\mathcal{L}_{\max}$ to the set $D_{\min}$, then we obtain the *minimal operator* $\mathcal{L}_{\min}$.

Let $y, z \in D_{\max}$. Then the q-Green formula of these functions is given by

$$\int_0^a \left[(\Upsilon y)(x)\overline{z(x)} - y(x)\overline{(\Upsilon z)(x)} \right] d_q x$$

$$= [y,z](a) - [y,z](c+) + [y,z](c-) - [y,z](0), \tag{6}$$

where

$$[y,z] := y(\overline{D_{q^{-1}} z}) - (D_{q^{-1}} y)\overline{z}.$$

Let us consider the operator $\mathcal{L}$ with a domain D consisting of vectors $y \in D_{\max}$, $(\mathcal{L}y = \Upsilon(y))$ that satisfy the boundary conditions (2) - (5).

Theorem 1. *The operator $\mathcal{L}$ is symmetric.*

Proof. Let $y, z \in D$. Then we have

$$\langle \mathcal{L}y, z\rangle_H - \langle y, \mathcal{L}z\rangle_H = \int_0^d (\Upsilon y)(x)\overline{z(x)} d_q x + \alpha \int_d^a (\Upsilon y)(x)\overline{z(x)} d_q x$$

$$-\int_0^d y(x)\overline{\Upsilon(z)(x)}d_qx-\alpha\int_d^a y(x)\overline{\Upsilon(z)(x)}d_qx.$$

From (6), we find

$$\langle\mathcal{L}y,z\rangle_H-\langle y,\mathcal{L}z\rangle_H=\alpha[y,z](a)-\alpha[y,z](c+)+[y,z](c-)-[y,z](0).$$

By conditions (2) - (5), we see that

$$\langle\mathcal{L}y,z\rangle_H=\langle y,\mathcal{L}z\rangle_H, \tag{7}$$

i.e., $\mathcal{L}$ is the symmetric operator. □

Corollary 2. *All eigenvalues of the problem (1) - (5) are real.*

Proof. Let μ be an eigenvalue with an eigenfunction φ. From (7), we find

$$\langle\mathcal{L}\varphi,\varphi\rangle_H=\langle\varphi,\mathcal{L}\varphi\rangle_H=\langle\varphi,\mu\varphi\rangle_H=\overline{\mu}\langle\varphi,\varphi\rangle_H. \tag{8}$$

On the other hand,

$$\langle\mathcal{L}\varphi,\varphi\rangle_H=\langle\mu\varphi,\varphi\rangle_H=\mu\langle\varphi,\varphi\rangle_H. \tag{9}$$

Combinig (8) and (9), we see that

$$\mu\langle\varphi,\varphi\rangle_H=\overline{\mu}\langle\varphi,\varphi\rangle_H,$$

$$(\mu-\overline{\mu})\langle\varphi,\varphi\rangle_H=0.$$

Hence

$$\mu=\overline{\mu}$$

since $\varphi\neq 0$. □

Corollary 3. *If ξ_1 and ξ_2 are two different eigenvalues of the problem defined by (1) - (5), then the corresponding eigenfunctions y_1 and y_2 are orthogonal.*

Proof. Let μ_1 and μ_2 be two different real eigenvalues with corresponding eigenfunctions φ_1 and φ_2, respectively. By (7), we obtain

$$\langle\mathcal{L}\varphi_1,\varphi_2\rangle_H=\langle\varphi_1,\mathcal{L}\varphi_2\rangle_H,$$

$$\langle\mu_1\varphi_1,\varphi_2\rangle_H=\langle\varphi_1,\mu_2\varphi_2\rangle_H,,$$

$$(\mu_1-\mu_2)\langle\varphi_1,\varphi_2\rangle_H=0.$$

Hence we see that φ_1 and φ_2 are orthogonal in H due to $\mu_1\neq\mu_2$. □

4. THE EXISTENCE THEOREM

Theorem 4. *For any $\lambda \in \mathbb{C}$, Eq. (1) has a solution $\varphi(\zeta,\lambda)$ satisfying conditions (2) - (4) which is an entire function of λ for every $\zeta \in [0,d) \cup (d,a]$.*

Proof. From [**2**], we conclude that the following problem

$$\left[-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q + v(\zeta)\right] y(\zeta) = \lambda y(\zeta),\ \zeta \in [0,d),$$

$$y(0) = -k_1,\ \mathcal{D}_{q^{-1}} y(0) = 1,$$

has a unique solution $\varphi_1(\zeta,\lambda)$ which is an entire function of λ.

Now let us consider the following problem

$$\left[-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q + v(\zeta)\right] y(\zeta) = \lambda y(\zeta),\ \zeta \in (d,a], \tag{10}$$

$$y(d+) = \frac{1}{k_2}\varphi_1(d-,\lambda), \tag{11}$$

$$\mathcal{D}_{q^{-1}} y(d+) = \frac{1}{k_3}\mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda). \tag{12}$$

$$u_n(\zeta,\lambda) = u_0(\zeta,\lambda)$$

Let

$$+q\int_d^{\zeta} \begin{pmatrix} \frac{\sin(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} \cos\left(\sqrt{\lambda}q\gamma;q\right) \\ -\cos\left(\sqrt{\lambda}\zeta;q\right)\frac{\sin(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} v(q\gamma)\, u_{n-1}(q\gamma,\lambda)\, d_q\gamma, \tag{13}$$

where

$$u_0(\zeta,\lambda) = \frac{1}{k_2}\varphi_1(d-,\lambda) + \frac{1}{k_3}(\zeta-d)\,\mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda),\ \zeta \in (d,a],$$

and the functions $\frac{\sin(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}}$, $\cos\left(\sqrt{\lambda}q\zeta;q\right)$ are the fundamental solutions of the equation

$$-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q y(\zeta) = \lambda y(\zeta). \tag{14}$$

It is obvious that the functions u_n are entire functions.

Let $\lambda \in \mathbb{C}$ be fixed. There exist positive numbers $\sigma(\lambda)$, $\widetilde{\sigma(\lambda)}$ and A such that

$$\left| \begin{pmatrix} \frac{\sin\left(\sqrt{\lambda}\zeta;q\right)}{\sqrt{\lambda}} \cos\left(\sqrt{\lambda}q\gamma;q\right) \\ -\cos\left(\sqrt{\lambda}\zeta;q\right) \frac{\sin\left(\sqrt{\lambda}q\gamma;q\right)}{\sqrt{\lambda}} \end{pmatrix} \right| \leq \sigma(\lambda),$$

$$\max_{\zeta \in (d,a]} |v(\zeta)| = A, |u_0(\zeta,\lambda)| \leq \widetilde{\sigma(\lambda)},\ \zeta \in (d,a].$$

Then, we have

$$\begin{aligned} &|u_1(\zeta,\lambda) - u_0(\zeta,\lambda)| \\ &\leq \left| q\int_d^\zeta \begin{pmatrix} \frac{\sin\left(\sqrt{\lambda}\zeta;q\right)}{\sqrt{\lambda}} \cos\left(\sqrt{\lambda}q\gamma;q\right) \\ -\cos\left(\sqrt{\lambda}\zeta;q\right) \frac{\sin\left(\sqrt{\lambda}q\gamma;q\right)}{\sqrt{\lambda}} \end{pmatrix} v(q\gamma)\, u_0(q\gamma,\lambda)\, d_q\gamma \right| \\ &\leq q\sigma(\lambda)\, A\widetilde{\sigma(\lambda)} \left| \int_0^\zeta d_q\gamma \right| = q\sigma(\lambda)\, A\widetilde{\sigma(\lambda)} \frac{\zeta(1-q)}{(1-q)}. \end{aligned}$$

Similarly, we obtain

$$|u_2(\zeta,\lambda) - u_1(\zeta,\lambda)| \leq q^2 \widetilde{\sigma(\lambda)} \frac{A^2\sigma^2(\lambda)\,\zeta^2(1-q)^2}{(1-q)(1-q^2)}.$$

It is easy to show that

$$(15) \qquad |u_{n+1}(\zeta,\lambda) - u_n(\zeta,\lambda)| \leq q^{n+1}\widetilde{\sigma(\lambda)} \frac{(A\sigma(\lambda)\,\zeta(1-q))^n}{(q;q)_n}\ (n = 1,2,...).$$

Thus, the series

$$(16) \qquad u_1(\zeta,\lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta,\lambda) - u_n(\zeta,\lambda)\}$$

is uniformly convergent with respect to variable ζ on $(d,a]$, due to the series

$$\sum_{n=1}^{\infty} q^{n+1}\widetilde{\sigma(\lambda)} \frac{(A\sigma(\lambda)\,\zeta(1-q))^n}{(q;q)_n}$$

is convergent.

If we define the function $\varphi_2(\zeta,\lambda)$ by the formula

$$\varphi_2(\zeta,\lambda) = u_1(\zeta,\lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta,\lambda) - u_n(\zeta,\lambda)\},$$

then we have

$$\lim_{n\to\infty} u_n(\zeta,\lambda) = \varphi_2(\zeta,\lambda).$$

From (13), we get

$$\mathcal{D}_q u_{n+1}(\zeta,\lambda) - \mathcal{D}_q u_n(\zeta,\lambda)$$

$$= q\int_d^{\zeta} \begin{pmatrix} \mathcal{D}_q \frac{\sin(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} \cos\left(\sqrt{\lambda}q\gamma;q\right) \\ -\mathcal{D}_q \cos\left(\sqrt{\lambda}\zeta;q\right) \frac{\sin(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \times$$
$$\times\, v(q\gamma) \begin{bmatrix} u_n(q\gamma,\lambda) \\ -u_{n-1}(q\gamma,\lambda) \end{bmatrix} d_q\gamma,$$

and

$$-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_{n+1}(\zeta,\lambda) + \frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_n(\zeta,\lambda)$$

$$= q\int_d^{\zeta} \begin{pmatrix} -\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q \frac{\sin(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} \cos\left(\sqrt{\lambda}q\gamma;q\right) \\ +\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q \cos\left(\sqrt{\lambda}\zeta;q\right) \frac{\sin(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \times$$
$$\times\, v(q\gamma) \begin{bmatrix} u_n(q\gamma,\lambda) \\ -u_{n-1}(q\gamma,\lambda) \end{bmatrix} d_q\gamma$$

$$-\, v(\zeta)\left[u_n(\zeta,\lambda) - u_{n-1}(\zeta,\lambda)\right].$$

By (15), the series

$$\sum_{n=1}^{\infty}\left(\mathcal{D}_q u_{n+1}(\zeta,\lambda) - \mathcal{D}_q u_n(\zeta,\lambda)\right)$$

and

$$\sum_{n=1}^{\infty}\left(-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_{n+1}(\zeta,\lambda) + \frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_n(\zeta,\lambda)\right)$$

are uniformly convergent on $(d,a]$ with respect to variable ζ for every $\lambda \in \mathbb{C}$. Hence, by (14), we obtain

$$-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q \varphi_2(\zeta,\lambda)$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_{n+1}(\zeta,\lambda) + \frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q u_n(\zeta,\lambda)\right)$$

$$= (\lambda - v(\zeta))\sum_{n=1}^{\infty}\left(u_n(\zeta,\lambda) - u_{n-1}(\zeta,\lambda)\right) = (\lambda - v(\zeta))\,\varphi_2(\zeta,\lambda).$$

It is easy to see that φ_2 satisfies (11) - (12). Therefore, we conclude that the function

$$\varphi(\zeta,\lambda)=\begin{cases}\varphi_1(\zeta,\lambda), & \zeta\in[0,d)\\ \varphi_2(\zeta,\lambda), & \zeta\in(d,a]\end{cases} \tag{17}$$

satisfies the problem (1) - (4). □

Similarly, one can obtain the following theorem.

Theorem 5. *For any $\lambda\in\mathbb{C}$, Eq. (1) has a solution*

$$\chi(\zeta,\lambda)=\begin{cases}\chi_1(\zeta,\lambda), & \zeta\in[0,d)\\ \chi_2(\zeta,\lambda), & \zeta\in(d,a]\end{cases} \tag{18}$$

satisfying conditions (3) - (5) which is an entire function of λ for every $\zeta\in[0,d)\cup(d,a]$.

5. THE CHARACTERISTIC FUNCTION

Now, we can define the following entire functions

$$\omega_1(\lambda)=W_q(\varphi_1,\chi_1)(\zeta),\ \omega_2(\lambda)=W_q(\varphi_2,\chi_2)(\zeta),$$

due to these Wronskians are independent of ζ for $\zeta\in[0,d)$ and $\zeta\in(d,a]$, respectively. By (3) - (4), we see that

$$\omega_1(\lambda)=\alpha\omega_2(\lambda).$$

Thus, the *characteristic function* of problem (1) - (5) is defined by the formula

$$\omega(\lambda):=\omega_1(\lambda)=\alpha\omega_2(\lambda).$$

Lemma 6. *Let*

$$\Delta(\lambda):=\begin{vmatrix}\Upsilon_1\varphi_1 & \Upsilon_1\chi_1 & \Upsilon_1\varphi_2 & \Upsilon_1\chi_2\\ \Upsilon_2\varphi_1 & \Upsilon_2\chi_1 & \Upsilon_2\varphi_2 & \Upsilon_2\chi_2\\ \Upsilon_3\varphi_1 & \Upsilon_3\chi_1 & \Upsilon_3\varphi_2 & \Upsilon_3\chi_2\\ \Upsilon_4\varphi_1 & \Upsilon_4\chi_1 & \Upsilon_4\varphi_2 & \Upsilon_4\chi_2\end{vmatrix},$$

where

$$\Upsilon_1 y:=y(0)+k_1\mathcal{D}_{q^{-1}}y(0),$$

$$\Upsilon_2 y:=y(a)+k_4\mathcal{D}_{q^{-1}}y(a),$$

$$\Upsilon_3 y:=y(d-)-k_2 y(d+),$$

$$\Upsilon_4 y:=\mathcal{D}_{q^{-1}}y(d-)-k_3\mathcal{D}_{q^{-1}}y(d+).$$

Then, for every $\lambda \in \mathbb{C}$, *we obtain*

$$\Delta(\lambda) = -\frac{1}{\alpha}\omega^3(\lambda).$$

Proof. From (17) and (18), we get

$$\Delta(\lambda)$$

$$= \begin{vmatrix} 0 & \omega_1(\lambda) & 0 & 0 \\ 0 & 0 & -\omega_2(\lambda) & 0 \\ \varphi_1(d-,\lambda) & \chi_1(d-,\lambda) & -k_2\varphi_2(d+,\lambda) & -k_2\chi_2(d+,\lambda) \\ \mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda) & \mathcal{D}_{q^{-1}}\chi_1(d-,\lambda) & -k_3\mathcal{D}_{q^{-1}}\varphi_2(d+,\lambda) & -k_3\mathcal{D}_{q^{-1}}\chi_2(d+,\lambda) \end{vmatrix}$$

$$= \omega_1(\lambda)\begin{vmatrix} 0 & -\omega_2(\lambda) & 0 \\ \varphi_1(d-,\lambda) & -k_2\varphi_2(d+,\lambda) & -k_2\chi_2(d+,\lambda) \\ \mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda) & -k_3\mathcal{D}_{q^{-1}}\varphi_2(d+,\lambda) & -k_3\mathcal{D}_{q^{-1}}\chi_2(d+,\lambda) \end{vmatrix}$$

$$= \omega_1(\lambda)\omega_2(\lambda)\begin{vmatrix} \varphi_1(d-,\lambda) & -k_2\chi_2(d+,\lambda) \\ \mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda) & -k_3\mathcal{D}_{q^{-1}}\chi_2(d+,\lambda) \end{vmatrix}$$

$$= -\omega_1(\lambda)\omega_2(\lambda)\begin{vmatrix} \varphi_1(d-,\lambda) & \chi_1(d-,\lambda) \\ \mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda) & \mathcal{D}_{q^{-1}}\chi_1(d-,\lambda) \end{vmatrix}$$

$$= -\omega_1^2(\lambda)\omega_2(\lambda) = -\frac{1}{k_2k_3}\omega^3(\lambda).$$

□

Theorem 7. *The eigenvalues of (1) - (5) same as the zeros of the entire function* $\omega(\lambda)$. *Hence the eigenvalues of (1) - (5) form a finite or infinite sequence without a finite accumulation point.*

Proof. Let $\lambda^{(0)}$ be a zero of $\omega(\lambda)$. Then $\omega_2\left(\lambda^{(0)}\right) = W_q(\varphi_2,\chi_2) = 0$, i.e., $\varphi_2 = \xi\chi_2$ for some $\xi \neq 0$. Thus φ_2 satisfies *(5). Therefore the function*

$$\varphi\left(\zeta,\lambda^{(0)}\right) = \begin{cases} \varphi_1\left(\zeta,\lambda^{(0)}\right), & \zeta \in [0,d) \\ \varphi_2\left(\zeta,\lambda^{(0)}\right). & \zeta \in (d,a] \end{cases}$$

satisfies (1) - (5), *i.e.,* $\lambda^{(0)}$ is an eigenvalue.

Let $\lambda^{(0)}$ be an eigenvalue and $\eta\left(\zeta,\lambda^{(0)}\right)$ be any corresponding eigenfunction. We want to show that $\omega\left(\lambda^{(0)}\right) = 0$. Assume that $\omega\left(\lambda^{(0)}\right) \neq 0$. Then we see that

$\omega_1\left(\lambda^{(0)}\right) \neq 0$ and $\omega_2\left(\lambda^{(0)}\right) \neq 0$. Thus there exist constants $\xi_i,\ i = 1, 2, 3, 4$, at least one of which is not zero, such that

$$\eta\left(\zeta, \lambda^{(0)}\right) = \begin{cases} \xi_1 \varphi_1\left(\zeta, \lambda^{(0)}\right) + \xi_2 \chi_1\left(\zeta, \lambda^{(0)}\right), & \zeta \in [0, d) \\ \xi_3 \varphi_2\left(\zeta, \lambda^{(0)}\right) + \xi_4 \chi_2\left(\zeta, \lambda^{(0)}\right), & \zeta \in (d, a]. \end{cases}$$

Consequently,

$$\Upsilon_i \eta\left(\zeta, \lambda^{(0)}\right) = 0,\ i = 1, 2, 3, 4,$$

due to $\eta\left(\zeta, \lambda^{(0)}\right)$ is the eigenfunction. So, we obtain

$$\det\left(\Upsilon_i \eta\left(\zeta, \lambda^{(0)}\right)\right) = \Delta(\lambda) = 0,$$

because at least one of the constants $\zeta_i,\ i = 1, 2, 3, 4$ is not zero. But, by Lemma 6, we see that $\Delta(\lambda) \neq 0$, a contradiction. □

6. GREEN'S FUNCTION

Let us consider the following problem

$$\left[-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q + \{-\lambda + v(\zeta)\}\right] y(\zeta)$$

$$= f(\zeta),\ \zeta \in [0, d) \cup (d, a],\ \lambda \in \mathbb{C},\ f \in H, \tag{19}$$

which satisfies (2) - (5).

By applying a q-analogue of the methods of variation of the constants, the general solution of (19) can be given by

$$\eta(\zeta, \lambda) = \begin{cases} \xi_1(\zeta, \lambda)\varphi_1(\zeta, \lambda) + \xi_2(\zeta, \lambda)\chi_1(\zeta, \lambda),\ \zeta \in [0, d) \\ \xi_3(\zeta, \lambda)\varphi_2(\zeta, \lambda) + \xi_4(\zeta, \lambda)\chi_2(\zeta, \lambda),\ \zeta \in (d, a], \end{cases}$$

where

$$\mathcal{D}_q \xi_1(\zeta, \lambda) = \frac{q}{\omega(\lambda)} f(q\zeta)\chi_1(q\zeta, \lambda),\ \zeta \in [0, d), \tag{20}$$

$$\mathcal{D}_q \xi_2(\zeta, \lambda) = -\frac{q}{\omega(\lambda)} f(q\zeta)\varphi_1(q\zeta, \lambda),\ \zeta \in [0, d), \tag{21}$$

$$\mathcal{D}_q \xi_3(\zeta, \lambda) = \frac{q}{\omega(\lambda)} f(q\zeta)\chi_2(q\zeta, \lambda),\ \zeta \in (d, a], \tag{22}$$

$$\mathcal{D}_q \xi_4(\zeta, \lambda) = -\frac{q}{\omega(\lambda)} f(q\zeta)\varphi_2(q\zeta, \lambda),\ \zeta \in (d, a]. \tag{23}$$

From (20) - (23), we obtain

$$\xi_1(\zeta,\lambda)=\frac{q}{\omega(\lambda)}\int_\zeta^d f(q\gamma)\chi_1(q\gamma,\lambda)\,d_q\gamma+\xi_1,\ \zeta\in[0,d),$$

$$\xi_2(\zeta,\lambda)=\frac{q}{\omega(\lambda)}\int_0^\zeta f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma+\xi_2,\ \zeta\in[0,d),$$

$$\xi_3(\zeta,\lambda)=\frac{q}{\omega(\lambda)}\int_\zeta^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma+\xi_3,\ \zeta\in(d,a],$$

$$\xi_4(\zeta,\lambda)=\frac{q}{\omega(\lambda)}\int_d^\zeta f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma+\xi_4,\ \zeta\in(d,a],$$

where ζ_i $(i=1,2,3,4)$ is an arbitrary constant. Thus we get

$$\eta(\zeta,\lambda)=\begin{cases}\xi_1\varphi_1(\zeta,\lambda)+\xi_2\chi_1(\zeta,\lambda)\\ +\frac{q}{\omega(\lambda)}\chi_1(\zeta,\lambda)\int_0^\zeta f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma\\ +\frac{q}{\omega(\lambda)}\varphi_1(\zeta,\lambda)\int_\zeta^d f(q\gamma)\chi_1(q\gamma,\lambda)\,d_q\gamma,\ \zeta\in[0,d)\\ \xi_3\varphi_2(\zeta,\lambda)+\xi_4\chi_2(\zeta,\lambda)\\ +\frac{q}{\omega(\lambda)}\varphi_2(\zeta,\lambda)\int_\zeta^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma\\ +\frac{q}{\omega(\lambda)}\chi_2(\zeta,\lambda)\int_d^\zeta f(q\gamma)\varphi_2(q\gamma,\lambda)\,d_q\gamma,\ \zeta\in(d,a],\end{cases}\tag{24}$$

where ζ_i $(i=1,2,3,4)$ is an arbitrary constant. From (24), we have

$$\mathcal{D}_{q^{-1}}\eta(\zeta,\lambda)=\begin{cases}\xi_1\mathcal{D}_{q^{-1}}\varphi_1(\zeta,\lambda)+\xi_2\mathcal{D}_{q^{-1}}\chi_1(\zeta,\lambda)\\ +\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\chi_1(\zeta,\lambda)\int_0^\zeta f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma\\ +\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\varphi_1(\zeta,\lambda)\int_\zeta^d f(q\gamma)\chi_1(q\gamma,\lambda)\,d_q\gamma,\ \zeta\in[0,d)\\ \xi_3\mathcal{D}_{q^{-1}}\varphi_2(\zeta,\lambda)+\xi_4\mathcal{D}_{q^{-1}}\chi_2(\zeta,\lambda)\\ +\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\varphi_2(\zeta,\lambda)\int_\zeta^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma\\ +\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\chi_2(\zeta,\lambda)\int_d^\zeta f(q\gamma)\varphi_2(q\gamma,\lambda)\,d_q\gamma,\ \zeta\in(d,a].\end{cases}$$

Hence

$$\Upsilon_1\eta = \eta(0) + k_1\mathcal{D}_{q^{-1}}\eta(0) = \xi_1\left[\varphi_1(0,\lambda) + k_1\mathcal{D}_{q^{-1}}\varphi_1(0,\lambda)\right]$$

$$+\xi_2\left[\chi_1(0,\lambda) + k_1\chi_1\mathcal{D}_{q^{-1}}(0,\lambda)\right]$$

$$+\frac{q}{\omega(\lambda)}\left[\varphi_1(0,\lambda) + k_1\mathcal{D}_{q^{-1}}\varphi_1(0,\lambda)\right]\int_0^d f(q\gamma)\chi_1(q\gamma,\lambda)\,d_q\gamma,$$

Since

$$\varphi_1(0,\lambda) + k_1\mathcal{D}_{q^{-1}}\varphi_1(0,\lambda) = 0$$

and

$$\chi_1(0,\lambda) + k_1\chi_1\mathcal{D}_{q^{-1}}(0,\lambda) = \omega(\lambda) \neq 0$$

we conclude that

$$\xi_2 = 0.$$

Similarly, we get

$$\Upsilon_2\eta = \eta(a) + k_4\mathcal{D}_{q^{-1}}\eta(a) = \xi_3\left[\varphi_2(a,\lambda) + k_4\mathcal{D}_{q^{-1}}\varphi_2(a,\lambda)\right]$$

$$+\xi_4\left[\chi_2(a,\lambda) + k_4\mathcal{D}_{q^{-1}}\chi_2(a,\lambda)\right]$$

$$+\frac{q}{\omega(\lambda)}\left[\chi_2(a,\lambda) + k_4\mathcal{D}_{q^{-1}}\chi_2(a,\lambda)\right]\int_d^a f(q\gamma)\varphi_2(q\gamma,\lambda)\,d_q\gamma$$

By using the following relations

$$\chi_2(a,\lambda) + k_4\mathcal{D}_{q^{-1}}\chi_2(a,\lambda) = 0$$

$$\varphi_2(a,\lambda) + k_4\mathcal{D}_{q^{-1}}\varphi_2(a,\lambda) = \omega(\lambda) \neq 0$$

we obtain

$$\xi_3 = 0.$$

Similarly, we have

$$\Upsilon_3\eta = \eta(d-) - k_2\eta(d+)$$

$$= \xi_1\varphi_1(d-,\lambda) - k_2\xi_4\chi_2(d+,\lambda)$$

$$+\frac{q}{\omega(\lambda)}\chi_1(d-,\lambda)\int_0^d f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma$$

$$-k_2\frac{q}{\omega(\lambda)}\varphi_2(d+,\lambda)\int_d^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma$$

and

$$\Upsilon_4\eta = \mathcal{D}_{q^{-1}}\eta(d-) - k_3\mathcal{D}_{q^{-1}}\eta(d+) = \xi_1\mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda)$$

$$+\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\chi_1(d-,\lambda)\int_0^d f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma - k_3\xi_4\mathcal{D}_{q^{-1}}\chi_2(d+,\lambda)$$

$$-k_3\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\varphi_2(d+,\lambda)\int_d^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma.$$

By virtue of (3) and (4), we have

$$(25)\qquad \begin{cases} \xi_1\varphi_1(d-,\lambda) - k_2\xi_4\chi_2(d+,\lambda) \\ = k_2\frac{q}{\omega(\lambda)}\varphi_2(d+,\lambda)\int_d^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma \\ -\frac{q}{\omega(\lambda)}\chi_1(d-,\lambda)\int_0^d f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma \\ \xi_1\mathcal{D}_{q^{-1}}\varphi_1(d-,\lambda) - k_3\xi_4\mathcal{D}_{q^{-1}}\chi_2(d+,\lambda) \\ = k_3\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\varphi_2(d+,\lambda)\int_d^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma \\ -\frac{q}{\omega(\lambda)}\mathcal{D}_{q^{-1}}\chi_1(d-,\lambda)\int_0^d f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma. \end{cases}$$

From (25), we deduce that

$$\xi_1 = \frac{q}{\omega(\lambda)}\int_d^a f(q\gamma)\chi_2(q\gamma,\lambda)\,d_q\gamma$$

and

$$\xi_4 = \frac{q}{\omega(\lambda)}\int_0^d f(q\gamma)\varphi_1(q\gamma,\lambda)\,d_q\gamma.$$

Finally, we obtain

$$\eta(\zeta,\lambda) = \frac{1}{\omega(\lambda)}\chi(\zeta,\lambda)\int_0^\zeta f(\gamma)\varphi(\gamma,\lambda)\,d_q\gamma$$

$$+\frac{1}{\omega(\lambda)}\varphi(\zeta,\lambda)\int_\zeta^a f(\gamma)\chi(\gamma,\lambda)\,d_q\gamma,$$

i.e.,

$$\eta(\zeta,\lambda) = \int_0^a G(\zeta,\gamma,\lambda)\,f(\gamma)d_q\gamma,$$

where $G(\zeta,\gamma,\lambda)$ is the Green's function defined by

$$(26)\qquad G(\zeta,\gamma,\lambda) = \begin{cases} \frac{1}{\omega(\lambda)}\chi(\zeta,\lambda)\,\varphi(\gamma,\lambda), & 0\le\gamma\le\zeta\le a,\ \zeta\ne d,\ \gamma\ne d, \\ \frac{1}{\omega(\lambda)}\chi(\gamma,\lambda)\,\varphi(\zeta,\lambda), & 0\le\zeta\le\gamma\le a,\ \zeta\ne d,\ \gamma\ne d. \end{cases}$$

7. EIGENFUNCTION EXPANSION

Theorem 8. *Suppose that* $\lambda = 0$ *is not an eigenvalue of (1)-(5).* $G(\zeta,\gamma)$ $(\lambda = 0)$ *defined as (26) is a q-Hilbert–Schmidt kernel, i.e.,*

$$\int_0^d \int_0^d |G(\zeta,\gamma)|^2 d_q\zeta d_q\gamma < +\infty, \ \int_a^d \int_a^d |G(\zeta,\gamma)|^2 d_q\zeta d_q\gamma < +\infty.$$

Proof. By (26), we deduce that

$$\int_0^d d_q\zeta \int_0^d |G(\zeta,\gamma)|^2 d_q\gamma < +\infty, \ \int_d^a d_q\zeta \int_d^a |G(\zeta,\gamma)|^2 d_q\gamma < +\infty,$$

due to $\chi(.,\lambda), \varphi(.,\lambda) \in H$. Therefore, we get

$$\int_0^d \int_0^d |G(\zeta,\gamma)|^2 d_q\zeta d_q\gamma < +\infty, \ \int_a^d \int_a^d |G(\zeta,\gamma)|^2 d_q\zeta d_q\gamma < +\infty. \tag{27}$$

□

Theorem 9 ([**17**])**.** *Let*

$$A\{t_i\} = \{x_i\}, \ i \in \mathbb{N} := \{1,2,3,...\},$$

where

$$x_i = \sum_{k=1}^{\infty} \eta_{ik} t_k, \ i,k \in \mathbb{N}. \tag{28}$$

If

$$\sum_{i,k=1}^{\infty} |\eta_{ik}|^2 < +\infty, \tag{29}$$

then the operator A *is compact in* l^2.

Theorem 10. *Let* $\mathcal{T}$ *be the integral operator* $\mathcal{T} : H \to H$,

$$f(\zeta) = \begin{cases} f^{(1)}(\zeta), \ \zeta \in [0,d) \\ f^{(2)}(\zeta), \ \zeta \in (d,a], \end{cases}$$

$$(\mathcal{T}f)(\zeta) = \begin{cases} \int_0^d G(\zeta,\gamma) f^{(1)}(\gamma) d_q\gamma, \ \zeta \in [0,d) \\ \\ \int_d^a G(\zeta,\gamma) f^{(2)}(\gamma) d_q\gamma, \ \zeta \in (d,a]. \end{cases}$$

Then $\mathcal{T}$ *is a self-adjoint and compact operator in space* H.

Proof. Let

$$\phi_i = \phi_i(\zeta) = \begin{cases} \phi_i^{(1)}(\zeta), & \zeta \in [0, d) \\ \phi_i^{(2)}(\zeta), & \zeta \in (d, a] \end{cases} \quad (i \in \mathbb{N})$$

be a complete, orthonormal basis of H. Let $i, k \in \mathbb{N}$. If we set

$$t_i = \langle f, \phi_i \rangle_H = \int_0^d f^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_q\zeta$$

$$+ \alpha \int_d^a f^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_q\zeta,$$

$$x_i = \langle g, \phi_i \rangle_H = \int_0^d g^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_q\zeta$$

$$+ \alpha \int_d^a g^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_q\zeta,$$

$$\eta_{ik} = \int_0^d \int_0^d G(\zeta, \gamma) \overline{\phi_i^{(1)}(\zeta) \phi_k^{(1)}(\gamma)} d_q\zeta d_q\gamma$$

$$+ \alpha \int_d^a \int_d^a G(\zeta, \gamma) \overline{\phi_i^{(2)}(\zeta) \phi_k^{(2)}(\gamma)} d_q\zeta d_q\gamma,$$

then H is mapped isometrically on to l^2. By this mapping, $\mathcal{T}$ transforms into the operator A defined by (28) in l^2 and (27) is translated into (29). It follows from Theorems 8 and 9 that A and $\mathcal{T}$ is compact.

Let $h, g \in H$. Then we have

$$\langle \mathcal{T}h, g\rangle_H = \int_0^d (\mathcal{T}h^{(1)})(\zeta)\overline{g^{(1)}(\zeta)}d_q\zeta + \alpha \int_d^a (\mathcal{T}h^{(2)})(\zeta)\overline{g^{(2)}(\zeta)}d_q\zeta$$

$$= \int_0^d \int_0^d G(\zeta,\gamma)\, h^{(1)}(\gamma) d_q\gamma \overline{g^{(1)}(\zeta)}d_q\zeta$$

$$+ \alpha \int_d^a \int_d^a G(\zeta,\gamma)\, h^{(2)}(\gamma) d_q\gamma \overline{g^{(2)}(\zeta)}d_q\zeta$$

$$= \int_0^d h^{(1)}(\gamma) \left(\int_0^d G(\gamma,\zeta)\, \overline{g^{(1)}(\zeta)} d_q\zeta \right) d_q\gamma$$

$$+ \alpha \int_d^a h^{(2)}(\gamma) \left(\int_d^a G(\gamma,\zeta)\, \overline{g^{(2)}(\zeta)} d_q\zeta \right) d_q\gamma = \langle h, \mathcal{T}g\rangle_H.$$

since $G(\zeta,\gamma)$ is a symmetric function. □

Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Then, $\ker \mathcal{L} = \{0\}$ and $\mathcal{T} = \mathcal{L}^{-1}$.

Theorem 11. *The operator $\mathcal{L}$ has an infinite countable set $\{\lambda_n\}_{n\in\mathbb{N}}$ of real eigenvalues which can be ordered as*

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_n| < ...,\ |\lambda_n| \to \infty \text{ as } n\to\infty.$$

The set of all normalized eigenfunctions of $\mathcal{L}$ forms an orthonormal basis for the space H and for $z \in H$, $\mathcal{T}z = h$, $\mathcal{L}h = z$, $\mathcal{L}\chi_n = \lambda_n\chi_n$ $(n \in \mathbb{N})$ the eigenfunction expansion formula

$$\mathcal{L}h = \sum_{n=1}^{\infty} \lambda_n \langle h, \chi_n\rangle_H \chi_n$$

is valid.

Proof. From the Hilbert–Schmidt theorem and the above theorem, we deduce that $\mathcal{T}$ has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \xi_n = 0.$$

Hence

$$|\lambda_n| = \frac{1}{|\xi_n|} \to \infty,\ n \to \infty.$$

Let $\{\chi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Then, for $z \in H$, we have $\mathcal{T}z = h$, $\mathcal{L}h = z$, $\mathcal{L}\chi_n = \lambda_n \chi_n$ $(n \in \mathbb{N})$ and

$$\mathcal{L}h = z = \sum_{n=1}^{\infty} \langle z, \chi_n \rangle_H \chi_n = \sum_{n=1}^{\infty} \langle \mathcal{L}h, \chi_n \rangle_H \chi_n$$

$$= \sum_{n=1}^{\infty} \langle h, \mathcal{L}\chi_n \rangle_H \chi_n = \sum_{n=1}^{\infty} \lambda_n \langle h, \chi_n \rangle_H \chi_n.$$

□

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