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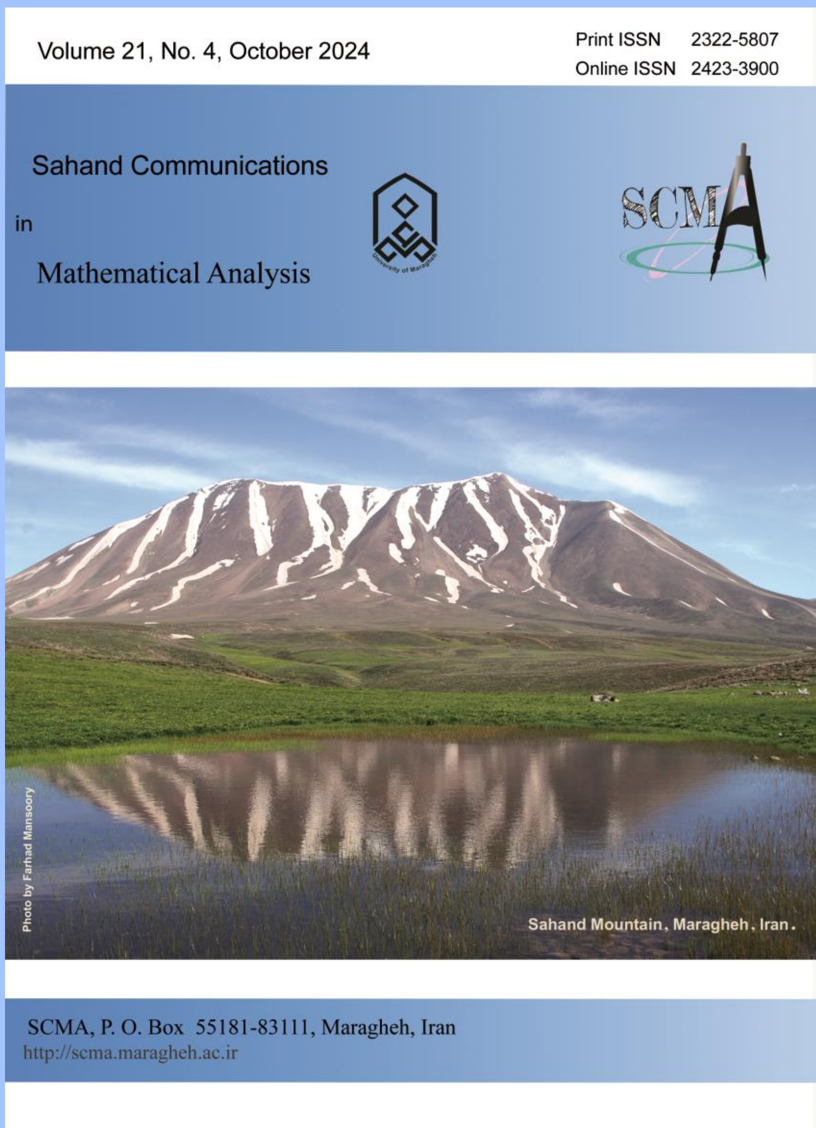

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Eigenfunction Expansions of Impulsive Dynamic Sturm-Liouville Problems

Bilender P. Allahverdiev[1], Hüseyin Tuna[2*] and Hamlet A. Isayev[3]

ABSTRACT. This paper studies impulsive dynamic Sturm-Liouville boundary value problems. The existence of a countably infinite set of eigenvalues and eigenfunctions is proved and a uniformly convergent expansion formula in the eigenfunctions is established.

1. INTRODUCTION

This paper deals with one-dimensional impulsive dynamic Sturm-Liouville problems

$$\left\{\begin{array}{l} -y^{\Delta\nabla}(\zeta)+q(\zeta)\,y(\zeta)=\lambda y(\zeta), \quad \zeta\in(a,d)\cup(d,b), \\ y(a)-h_1y^{\nabla}(a)=0, \\ y(d-)=\eta y(d+), \\ y^{\nabla}(d-)=\frac{1}{\eta}y^{\nabla}(d+), \\ y(b)+h_2y^{\nabla}(b)=0. \end{array}\right. \tag{1.1}$$

Here $\mathbb{T}$ is a time scale, $I_1:=[a,d)$, $I_2:=(d,b]$, $-\infty<a<d<b<+\infty$, $I:=I_1\cup I_2$, $I\subset\mathbb{T}$, λ is a complex eigenvalue parameter, $h_1>0$, $h_2>0$, $\eta>0$ and q is a real-valued continuous function on I. We will show that the boundary value problem (1.1) has a countably infinite set of eigenvalues and eigenfunctions. Moreover, a uniformly convergent eigenfunction expansion is obtained for this problem. Eigenfunction expansions are

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intensively researched not only for solving partial differential equations with the Fourier method, but also because they are important for spectral theory (see [1, 3, 4, 6–14]).

Guseinov [6] investigated the uniform convergence of eigenfunction expansions for the Sturm-Liouville problem defined as

$$\begin{aligned} &-y^{\Delta\nabla}(\zeta)=\lambda y(\zeta), \quad \zeta\in(a,b) \\ &y(a)=y(b)=0, \end{aligned}$$

on the time scale in 2007. Then Huseynov and Bairamov [7] obtained similar results for the following problem

$$\begin{aligned} &-\left[p(\zeta)\, y^{\Delta}(\zeta)\right]^{\nabla}+q(\zeta)\, y(\zeta)=\lambda y(\zeta), \quad \zeta\in(a,b), \\ &y(a)-h_1 y^{[\Delta]}(a)=0, \\ &y(b)+h_2 y^{[\Delta]}(b)=0. \end{aligned}$$

The difference in this work from the previous ones is that it deals with the problem of impulsive boundary conditions.

2. Main Results

Firstly, we assume that the reader is familiar with the basic facts of time scales (see [5]). Now, we consider the following boundary value problem (BVP)

$$\begin{aligned} (Ly)(\zeta) &:= -y^{\Delta\nabla}(\zeta)+q(\zeta)\, y(\zeta) \\ &= \lambda y(\zeta), \quad \zeta\in(a,d)\cup(d,b) \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} y(a)-h_1 y^{\nabla}(a)=0, \\ y(b)+h_2 y^{\nabla}(b)=0, \end{aligned} \tag{2.2}$$

and impulsive conditions

$$y(d-)=\eta y(d+), \tag{2.3}$$

$$y^{\nabla}(d-)=\frac{1}{\eta} y^{\nabla}(d+), \tag{2.4}$$

where $\mathbb{T}$ is the time scale, $h_1>0$, $h_2>0$, $\eta>0$, $I_1:=[a,d)$, $I_2:=(d,b]$, $-\infty<a<d<b<+\infty$, $I:=I_1\cup I_2$, $I\subset\mathbb{T}$, λ is a complex eigenvalue parameter, q is a real-valued continuous function on I, $d\in\mathbb{T}$ is a regular point for L, one-sided limits $q(d\pm)$ exist and are finite.

Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}, \quad t\in\mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad t \in \mathbb{T}.$$

If $\sigma(t) > t$, we say that t is right scattered, while if $\rho(t) < t$, we say that t is left scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Some function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there exists a number $f^{\Delta}(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ.

$H = L^2(I_1) \dot{+} L^2(I_2)$ is a Hilbert space endowed with the following inner product

$$\langle \psi, \omega \rangle := \int_a^d \psi^{(1)} \overline{\omega^{(1)}} \Delta\zeta + \int_d^b \psi^{(2)} \overline{\omega^{(2)}} \Delta\zeta,$$

where

$$\psi(\zeta) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in I_1 \\ \psi^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

and

$$\omega(\zeta) = \begin{cases} \omega^{(1)}(\zeta), & \zeta \in I_1 \\ \omega^{(2)}(\zeta), & \zeta \in I_2. \end{cases}$$

Let

$$\mathcal{T} : \mathcal{D} \subset H \to H, \qquad \mathcal{T}y = Ly, \quad y \in \mathcal{D},$$

where

$$\mathcal{D} = \left\{ y \in H : \begin{array}{l} y \in \Delta\text{-}AC(I), \\ y^{\Delta} \in \nabla\text{-}AC(I), \\ y(d\pm) \text{ exist and are finite}, \\ y(a) - h_1 y^{\nabla}(a) = 0, \\ y(b) + h_2 y^{\nabla}(b) = 0, \\ y(d-) = \eta y(d+), \\ y^{\nabla}(d-) = \frac{1}{\eta} y^{\nabla}(d+) \\ Ly \in H \end{array} \right\}.$$

Then, for $y, z \in \mathcal{D}$, we have

$$\int_a^b Ly\overline{z}\Delta\zeta - \int_a^b y\overline{Lz}\Delta\zeta = [y,z]_{d-} - [y,z]_a + [y,z]_b - [y,z]_{d+}, \tag{2.5}$$

where

$$[y,z]_\zeta = y(\zeta)z^{\nabla}(\zeta) - z^{\nabla}(\zeta)y(\zeta), \quad (\zeta \in I).$$

Theorem 2.1. *The operator $\mathcal{T}$ is a positive self-adjoint in H.*

Proof. Let $y, z \in \mathcal{D}$. It follows from conditions (2.2)-(2.4) and (2.5) that

$$\langle \mathcal{T} y, z\rangle = \langle y, \mathcal{T} z\rangle. \tag{2.6}$$

Since $\mathcal{D}$ is a dense subset in H, we see that $\mathcal{T}$ *is a self-adjoint operator.*

Let

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in I_1 \\ y^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \qquad y \in \mathcal{D}.$$

Then we conclude that

$$\begin{aligned}
\langle \mathcal{T} y, y\rangle &= \int_a^d \left[- \left[y^{(1)\Delta}(\zeta)\right]^{\nabla} + q(\zeta) y^{(1)}(\zeta)\right] \overline{y^{(1)}(\zeta)} \Delta\zeta \\
&\quad + \int_d^b \left[- \left[y^{(2)\Delta}(\zeta)\right]^{\nabla} + q(\zeta) y^{(2)}(\zeta)\right] \overline{y^{(2)}(\zeta)} \Delta\zeta \\
&= \int_a^d - \left[y^{(1)\Delta}(\zeta)\right]^{\nabla} \overline{y^{(1)}(\zeta)} \Delta\zeta + \int_a^d q(\zeta) \left| y^{(1)}(\zeta)\right|^2 \Delta\zeta \\
&\quad + \int_d^b - \left[y^{(2)\Delta}(\zeta)\right]^{\nabla} \overline{y^{(2)}(\zeta)} \Delta\zeta + \int_d^b q(\zeta) \left| y^{(2)}(\zeta)\right|^2 \Delta\zeta \\
&= - \int_a^d \left[y^{(1)\Delta}(\rho(\zeta))\right]^{\Delta} \overline{y(\zeta)} \Delta\zeta + \int_a^d q(\zeta) \left| y^{(1)}(\zeta)\right|^2 \Delta\zeta \\
&\quad - \int_d^b \left[y^{(2)\Delta}(\rho(\zeta))\right]^{\Delta} \overline{y^{(2)}(\zeta)} \Delta\zeta + \int_d^b q(\zeta) \left| y^{(2)}(\zeta)\right|^2 \Delta\zeta \\
&= - \left[y^{(1)\Delta}(\rho(d-)) \overline{y^{(1)}(d-)}\right] + \left[y^{(1)\Delta}(a) \overline{y^{(1)}(a)}\right] \\
&\quad - \left[y^{(2)\Delta}(\rho(b)) \overline{y^{(2)}(b)}\right] + \left[y^{(2)\Delta}(d+) \overline{y^{(2)}(d+)}\right] \\
&\quad + \int_a^d \left| y^{(1)\Delta}(\zeta)\right|^2 \Delta\zeta + \int_a^d q(\zeta) \left| y^{(1)}(\zeta)\right|^2 \Delta\zeta \\
&\quad + \int_d^b \left| y^{(2)\Delta}(\zeta)\right|^2 \Delta\zeta + \int_d^b q(\zeta) \left| y^{(2)}(\zeta)\right|^2 \Delta\zeta \\
&= h_1 \left| y^{(1)\Delta}(a)\right|^2 + h_2 \left| y^{(2)\Delta}(\rho(b))\right|^2 \\
&\quad + \int_a^d \left| y^{(1)\Delta}(\zeta)\right|^2 \Delta\zeta + \int_a^d q(\zeta) \left| y^{(1)}(\zeta)\right|^2 \Delta\zeta \\
&\quad + \int_d^b \left| y^{(2)\Delta}(\zeta)\right|^2 \Delta\zeta + \int_d^b q(\zeta) \left| y^{(2)}(\zeta)\right|^2 \Delta\zeta \\
&> 0,
\end{aligned}$$

since $h_1, h_2 > 0$ and $q(\zeta) \geq 0$ for $\zeta \in I$. $\square$

Let

$$u(\zeta) = \begin{cases} u^{(1)}(\zeta), & \zeta \in I_1 \\ u^{(2)}(\zeta), & \zeta \in I_2 \end{cases}$$

and

$$\chi(\zeta) = \begin{cases} \chi^{(1)}(\zeta), & \zeta \in I_1 \\ \chi^{(2)}(\zeta), & \zeta \in I_2 \end{cases}$$

be solutions of the problem

$$-\left[y^{\Delta}(\zeta)\right]^{\nabla} + q(\zeta)\, y(\zeta) = 0,$$

$$y(d-) = \eta y(d+), \qquad y^{\nabla}(d-) = \frac{1}{\eta} y^{\nabla}(d+),$$

satisfying

$$u^{(1)}(a) = h_1, \quad u^{(1)\nabla}(a) = 1,$$
$$\chi^{(2)}(b) = -h_2, \quad \chi^{(2)\nabla}(b) = 1.$$

Corollary 2.2. *Since $\mathcal{T}$ is the positive operator and positive operators have positive eigenvalues, zero is not an eigenvalue of $\mathcal{T}$.*

Definition 2.3 ([5])**.** The Δ-Wronskian of the functions y and z is defined by the formula

$$W_{\Delta}(y,z) := yz^{\Delta} - zy^{\Delta}.$$

Theorem 2.4. *Let*

(2.7)
$$G(\zeta,t) = -\frac{1}{W_{\Delta}(u,\chi)} \begin{cases} u(\zeta)\,\chi(t), & a \le \zeta \le t \le b,\ \zeta \ne d,\ t \ne d, \\ u(t)\,\chi(\zeta), & a \le t \le \zeta \le b,\ \zeta \ne d,\ t \ne d. \end{cases}$$

Then $G(\zeta,t)$ is a Δ-Hilbert–Schmidt kernel, i.e.,

$$\int_a^d \int_a^d |G(\zeta,t)|^2\, \Delta\zeta \Delta t < \infty, \qquad \int_d^b \int_d^b |G(\zeta,t)|^2 \Delta\zeta\Delta t < \infty.$$

Proof. By (2.7), we deduce that

$$\int_a^d \Delta\zeta \int_a^d |G(\zeta,t)|^2\, \Delta t < \infty, \qquad \int_d^b \Delta\zeta \int_d^b |G(\zeta,t)|^2\, \Delta t < \infty$$

since $u(.)\,\chi(.) \in H \times H$. Then, we find

(2.8)
$$\int_a^d \int_a^d |G(\zeta,t)|^2\, \Delta\zeta\Delta t < \infty,$$
$$\int_d^b \int_d^b |G(\zeta,t)|^2 \Delta\zeta\Delta t < \infty.$$

□

Theorem 2.5 ([10])**.** *Let A be an operator defined as*

$$A\{\zeta_i\} = \{y_i\},$$

where $i \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and

$$y_i = \sum_{k=1}^{\infty} a_{ik}\zeta_k. \tag{2.9}$$

If

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty \tag{2.10}$$

then A is compact in l^2.

Theorem 2.6. *Let $K : H \to H$ be an operator defined as*

$$(Kf)(\zeta) = \begin{cases} \int_a^d G(\zeta,\gamma) f^{(1)}(\gamma)\Delta\gamma, & \zeta \in [a,d), \\ \int_d^b G(\zeta,\gamma) f^{(2)}(\gamma)\Delta\gamma, & \zeta \in (d,b], \end{cases} \tag{2.11}$$

where

$$f(\zeta) = \begin{cases} f^{(1)}(\zeta), & \zeta \in [a,d), \\ f^{(2)}(\zeta), & \zeta \in (d,b], \end{cases} \quad y \in H.$$

Then K is a compact operator.

Proof. Let $\{\phi_i\}_{i\in\mathbb{N}}$, be a complete, orthonormal basis of H, where

$$\phi_i = \phi_i(\zeta) = \begin{cases} \phi_i^{(1)}(\zeta), & \zeta \in [a,d), \\ \phi_i^{(2)}(\zeta), & \zeta \in (d,b]. \end{cases}$$

Define

$$\begin{aligned} \zeta_i &= \langle f, \phi_i \rangle \\ &= \int_a^d f^{(1)}(\zeta)\overline{\phi_i^{(1)}(\zeta)}\Delta\zeta + \int_d^b f^{(2)}(\zeta)\overline{\phi_i^{(2)}(\zeta)}\Delta\zeta, \end{aligned}$$

and

$$\begin{aligned} y_i &= \langle g, \phi_i \rangle \\ &= \int_a^d g^{(1)}(\zeta)\overline{\phi_i^{(1)}(\zeta)}\Delta\zeta + \int_d^b g^{(2)}(\zeta)\overline{\phi_i^{(2)}(\zeta)}\Delta\zeta, \end{aligned}$$

also

$$\begin{aligned} a_{ik} &= \int_a^d \int_a^d G(\zeta,t)\overline{\phi_i^{(1)}(\zeta)\phi_k^{(1)}(t)}\Delta\zeta\Delta t \\ &\quad + \int_d^b \int_d^b G(\zeta,t)\overline{\phi_i^{(2)}(\zeta)\phi_k^{(2)}(t)}\Delta\zeta\Delta t, \quad i,k \in \mathbb{N}. \end{aligned}$$

Thus, H is mapped isometrically l^2. K transforms into A in l^2. By Theorem 2.5, we conclude that A is compact. Hence, K is compact. □

Since $K = \mathcal{T}^{-1}$, the completeness of the system of all eigenfunctions of $\mathcal{T}$ is equivalent to the completeness of the system of all eigenfunctions of K. From the Hilbert-Schmidt theorem, we obtain the following theorem.

Theorem 2.7. *For the BVP (2.1)-(2.4), there exists an orthonormal basis* $\{\psi_k\}_{k\in\mathbb{N}}$ *in* H. *For* $f \in H$, *we get*

$$f(\zeta) = \sum_{k=1}^{\infty} c_k \psi_k(\zeta), \tag{2.12}$$

where

$$c_k = \langle f, \psi_k \rangle, \quad k \in \mathbb{N}.$$

Thus, we have

$$\lim_{N\to\infty} \left\{ \begin{array}{c} \int_a^d \left| f^{(1)}(\zeta) - \sum\limits_{k=1}^{N} c_k \psi_k^{(1)}(\zeta) \right|^2 \Delta\zeta \\ + \int_d^b \left| f^{(2)}(\zeta) - \sum\limits_{k=1}^{N} c_k \psi_k^{(2)}(\zeta) \right|^2 \Delta\zeta \end{array} \right\} = 0, \tag{2.13}$$

Moreover, it follows from (2.13) that

$$\int_a^d \left| f^{(1)}(\zeta) \right|^2 \Delta\zeta + \int_d^b \left| f^{(2)}(\zeta) \right|^2 \Delta\zeta = \sum_{k=1}^{\infty} |c_k|^2. \tag{2.14}$$

The main result of the article is the following theorem.

Theorem 2.8. *Let* $f, f^{\nabla} : I \to \mathbb{R}$ *be continuous functions on* I, *one-sided limits* $f(d\pm)$, $f^{\nabla}(d\pm)$ *exist and are finite and satisfying (2.2)-(2.4). Then the series*

$$f(\zeta) = \sum_{k=1}^{\infty} c_k \psi_k(\zeta), \tag{2.15}$$

where

$$c_k = \langle f, \psi_k \rangle, \quad k \in \mathbb{N},$$

converges uniformly to f *on* I.

Proof. Let

$$\begin{aligned} S(y) := {} & h_1 \left| y^{(1)\Delta}(a) \right|^2 + h_2 \left| y^{(2)\Delta}(\rho(b)) \right|^2 \\ & + \int_a^d \left| y^{(1)\Delta}(\zeta) \right|^2 \Delta\zeta + \int_a^d q(\zeta) \left| y^{(1)}(\zeta) \right|^2 \Delta\zeta \\ & + \int_d^b \left| y^{(2)\Delta}(\zeta) \right|^2 \Delta\zeta + \int_d^b q(\zeta) \left| y^{(2)}(\zeta) \right|^2 \Delta\zeta, \end{aligned} \tag{2.16}$$

and $S(y) \geq 0$. If we take

$$y = f(\zeta) - \sum_{k=1}^{N} c_k \psi_k(\zeta)$$

in (2.16), we deduce that

$$\begin{aligned}
&S\left(f(\zeta) - \sum_{k=1}^{N} c_k \psi_k(\zeta)\right) \\
&\quad = h_1 \left[f^{(1)\Delta}(a) - \sum_{k=1}^{N} c_k \psi_k^{(1)\Delta}(a)\right]^2 + h_2 \\
&\qquad \times \left[f^{(2)\Delta}(\rho(b)) - \sum_{k=1}^{N} c_k \left(\psi_k^{(2)\Delta}(b)\right)\right]^2 \\
&\qquad + \int_a^d \left(f^{(1)\Delta}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(1)\Delta}(\zeta)\right)^2 \Delta\zeta \\
&\qquad + \int_d^b \left(f^{(2)\Delta}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(2)\Delta}(\zeta)\right)^2 \Delta\zeta \\
&\qquad + \int_a^b q(\zeta) \left(f^{(1)}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(1)}(\zeta)\right)^2 \Delta\zeta \\
&\qquad + \int_d^b q(\zeta) \left(f^{(2)}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(2)}(\zeta)\right)^2 \Delta\zeta \\
&\quad = h_1 \left[f^{(1)\Delta}(a)\right]^2 + h_2 \left[f^{(2)\Delta}(\rho(b))\right]^2 \\
&\qquad - 2 \sum_{k=1}^{N} c_k \begin{bmatrix} -h_1 f^{(1)\Delta}(a)\, \psi_k^{(1)\Delta}(a) \\ -h_2 f^{(2)\Delta}(\rho(b))\, \psi_k^{(2)\Delta}(b) \end{bmatrix} \\
&\qquad - \sum_{k,m=1}^{N} c_k c_m \begin{bmatrix} -h_1 \psi_k^{(1)\Delta}(a)\, \psi_m^{(1)\Delta}(a) \\ -h_2 \psi_k^{(2)\Delta}(b)\, \psi_m^{(2)\Delta}(b) \end{bmatrix} \\
&\qquad + \int_a^d (f^{(1)\Delta}(\zeta))^2 \Delta\zeta + \int_a^d q(\zeta)\, f^{(1)2}(\zeta)\, \Delta\zeta \\
&\qquad + \int_d^b (f^{(2)\Delta}(\zeta))^2 \Delta\zeta + \int_d^b q(\zeta)\, f^{(2)2}(\zeta)\, \Delta\zeta
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{k=1}^{N}c_k\left[\int_a^d f^{(1)\Delta}(\zeta)\,\psi_k^{(1)\Delta}(\zeta)\,\Delta\zeta+\int_d^b f^{(2)\Delta}(\zeta)\,\psi_k^{(2)\Delta}(\zeta)\,\Delta\zeta\right]\\
&-2\sum_{k=1}^{N}c_k\left[\begin{array}{c}\int_a^d q(\zeta)\,f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\,\Delta\zeta\\ +\int_d^b q(\zeta)\,f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\,\Delta\zeta\end{array}\right]\\
&+\sum_{k,m=1}^{N}c_kc_m\left[\begin{array}{c}\int_a^d \psi_k^{(1)\Delta}(\zeta)\,\psi_m^{(1)\Delta}(\zeta)\,\Delta\zeta\\ +\int_d^b \psi_k^{(2)\Delta}(\zeta)\,\psi_m^{(2)\Delta}(\zeta)\,\Delta\zeta\end{array}\right]\\
&+\sum_{k,m=1}^{N}c_kc_m\left[\begin{array}{c}\int_a^d q(\zeta)\,\psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\,\Delta\zeta\\ +\int_d^b q(\zeta)\,\psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\,\Delta\zeta\end{array}\right].
\end{aligned}
$$

Applications of (2.2)-(2.4) and integration by parts yield

$$
\begin{aligned}
&\int_a^d \psi_k^{(1)\Delta}(\zeta)\,f^{(1)\Delta}(\zeta)\,\Delta\zeta+\int_a^d q(\zeta)\,f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\,\Delta\zeta\\
&\quad+\int_d^b \psi_k^{(2)\Delta}(\zeta)\,f^{(2)\Delta}(\zeta)\,\Delta\zeta+\int_d^b q(\zeta)\,f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\,\Delta\zeta\\
&=\psi_k^{(1)\Delta}(\rho(d-))\,f^{(1)}(d-)-\psi_k^{(1)\Delta}(a)\,f^{(1)}(a)\\
&\quad+\psi_k^{(2)\Delta}(\rho(b))\,f^{(2)}(b)-\psi_k^{(2)\Delta}(d+)\,f^{(2)}(d+)\\
&\quad-\int_a^d f^{(1)}(\zeta)\left(\psi_k^{(1)\Delta}(\zeta)\right)^{\nabla}\Delta\zeta-\int_d^b f^{(2)}(\zeta)\left(\psi_k^{(2)\Delta}(\zeta)\right)^{\nabla}\Delta\zeta\\
&\quad+\int_a^d q(\zeta)\,f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\,\Delta\zeta+\int_d^b q(\zeta)\,f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\,\Delta\zeta\\
&=-h_2f^{(2)\nabla}(b)\,\psi_k^{(2)\nabla}(b)-h_1f^{(1)\nabla}(a)\,\psi_k^{(1)\Delta}(a)\\
&\quad+\int_a^d f^{(1)}(\zeta)\left[-\left(\psi_k^{(1)\Delta}(\zeta)\right)^{\nabla}+q(\zeta)\,\psi_k^{(1)}(\zeta)\right]\Delta\zeta\\
&\quad+\int_d^b f^{(2)}(\zeta)\left[-\left(\psi_k^{(2)\Delta}(\zeta)\right)^{\nabla}+q(\zeta)\,\psi_k^{(2)}(\zeta)\right]\Delta\zeta\\
&=-h_2f^{(2)\nabla}(b)\,\psi_k^{(2)\nabla}(b)-h_1f^{(1)\nabla}(a)\,\psi_k^{(1)\Delta}(a)+\lambda_kc_k,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_a^d \psi_k^{(1)\Delta}(\zeta)\,\psi_m^{(1)\Delta}(\zeta)\,\Delta\zeta+\int_d^b \psi_k^{(2)\Delta}(\zeta)\,\psi_m^{(2)\Delta}(\zeta)\,\Delta\zeta\\
&\quad+\int_a^d q(\zeta)\,\psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\,\Delta\zeta+\int_d^b q(\zeta)\,\psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\,\Delta\zeta\\
&=\psi_m^{(1)\Delta}(\rho(d-))\,\psi_k^{(1)}(d-)+\psi_m^{(2)\Delta}(\rho(b))\,\psi_k^{(2)}(b)
\end{aligned}
$$

$$
\begin{aligned}
&-\psi_m^{(1)\Delta}(a)\,\psi_k^{(1)}(a)-\psi_m^{(2)\Delta}(d+)\,\psi_k^{(2)}(d+)\\
&+\int_a^d \psi_k^{(1)}(\zeta)\left[-\left(\psi_m^{(1)\Delta}(\zeta)\right)^{\nabla}+q(\zeta)\,\psi_m^{(1)}(\zeta)\right]\Delta\zeta\\
&+\int_d^b \psi_k^{(2)}(\zeta)\left[-\left(\psi_m^{(2)\Delta}(\zeta)\right)^{\nabla}+q(\zeta)\,\psi_m^{(2)}(\zeta)\right]\Delta\zeta\\
=\;&\psi_k^{(2)}(b)\,\psi_m^{(2)\nabla}(b)-\psi_k^{(1)}(a)\,\psi_k^{(1)\nabla}(a)\\
&+\lambda_k\left[\int_a^d \psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\,\Delta\zeta+\int_d^b \psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\,\Delta\zeta\right]\\
=\;&-h_1\psi_k^{(2)\nabla}(b)\,\psi_m^{(2)\nabla}(b)-h_2\psi_k^{(1)\nabla}(a)\,\psi_m^{(1)\nabla}(a)+\lambda_k\delta_{km},
\end{aligned}
$$

where

$$
\delta_{km}:=\begin{cases}1, & \text{if } k=m,\\ 0, & \text{if } k\neq m.\end{cases}
$$

Therefore, we obtain

$$
\begin{aligned}
S\left(f(\zeta)-\sum_{k=1}^{N}c_k\psi_k(\zeta)\right)=\;&h_1\left[f^{(1)\Delta}(a)\right]^2+h_2\left[f^{(2)\Delta}(\rho(b))\right]^2\\
&+\int_a^d (f^{(1)\Delta}(\zeta))^2\Delta\zeta+\int_a^d q(\zeta)\,f^{(1)2}(\zeta)\,\Delta\zeta\\
&+\int_d^b (f^{(2)\Delta}(\zeta))^2\Delta\zeta+\int_d^b q(\zeta)\,f^{(2)2}(\zeta)\,\Delta\zeta\\
&-\sum_{k=1}^{N}\lambda_k c_k^2.
\end{aligned}
$$

Moreover, we find

$$
\begin{aligned}
\sum_{k=1}^{\infty}\lambda_k c_k^2\le\;&h_1\left[f^{(1)\Delta}(a)\right]^2+h_2\left[f^{(2)\Delta}(\rho(b))\right]^2\\
&+\int_a^d (f^{(1)\Delta}(\zeta))^2\Delta\zeta+\int_a^d q(\zeta)\,f^{(1)2}(\zeta)\,\Delta\zeta\\
&+\int_d^b (f^{(2)\Delta}(\zeta))^2\Delta\zeta+\int_d^b q(\zeta)\,f^{(2)2}(\zeta)\,\Delta\zeta.
\end{aligned}
\tag{2.17}
$$

since S is nonnegative for all N. Thus, the convergence of the series

$$
\sum_{k=1}^{\infty}\lambda_k c_k^2
$$

follows.

Now, we shall prove that the series

$$\sum_{k=1}^{\infty} |c_k \psi_k(\zeta)| \tag{2.18}$$

is uniformly convergent on I. Since $\mathcal{T}\psi_k = \lambda_k \psi_k$, $k \in \mathbb{N}$, we obtain

$$\psi_k(\zeta) = \lambda_k \left(\mathcal{T}^{-1}\psi_k\right)(\zeta) = \lambda_k \langle G(\zeta, t), \psi_k \rangle, \quad k \in \mathbb{N}.$$

If we rewrite the series (2.18), we see that

$$\sum_{k=1}^{\infty} |c_k \psi_k(\zeta)| = \sum_{k=1}^{\infty} \lambda_k |c_k \Upsilon_k(\zeta)|, \tag{2.19}$$

where

$$\Upsilon_k(\zeta) = \langle G(\zeta, t), \psi_k \rangle, \quad k \in \mathbb{N}.$$

This can be regarded as the Fourier coefficients of $G(\zeta, t)$ as a function of t. It follows from (2.17) that

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k \Upsilon_k^2(\zeta) \leq h_1 \left[G^{(1)\Delta}(\zeta, a)\right]^2 + h_2 \left[G^{(2)\Delta}(\zeta, \rho(b))\right]^2 \\ + \int_a^d (G^{(1)\Delta}(\zeta, t))^2 \Delta t + \int_a^d q(t) G^{(1)2}(\zeta, t) \Delta t \\ + \int_d^b (G^{(2)\Delta}(\zeta, t))^2 \Delta t + \int_a^b q(t) G^{(2)2}(\zeta, t) \Delta t.\end{aligned}$$

Since all the functions appearing under the integral sign are bounded, we infer that

$$\sum_{k=1}^{\infty} \lambda_k \Upsilon_k^2(\zeta) \leq C,$$

where C is a constant. Applying the Cauchy–Schwartz inequality to (2.19), we obtain

$$\begin{aligned}\sum_{k=n}^{n+m} \lambda_k |c_k \Upsilon_k(\zeta)| &\leq \sqrt{\sum_{k=n}^{n+m} \lambda_k c_k^2} \sqrt{\sum_{k=n}^{n+m} \lambda_k \Upsilon_k^2(\zeta)} \\ &\leq \sqrt{C} \sqrt{\sum_{k=n}^{n+m} \lambda_k c_k^2}.\end{aligned} \tag{2.20}$$

From (2.17) and (2.20), the series (2.18) is uniformly convergent on I. Since

$$\left|\sum_{k=1}^{\infty} c_k \psi_k(\zeta)\right| \leq \sum_{k=1}^{\infty} |c_k \psi_k(\zeta)|,$$

the series (2.15) is also uniformly convergent on I.

Let

$$f_1(\zeta) = \sum_{k=1}^{\infty} c_k \psi_k(\zeta). \tag{2.21}$$

Then, for $k \in \mathbb{N}$, we obtain

$$\int_a^d f_1^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\,\Delta\zeta + \int_d^b f_1^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\,\Delta\zeta = c_k$$

due to the series (2.21) is uniformly convergent on I. Therefore, the Fourier coefficients of f and f_1 are the same. From (2.14), we find $f - f_1 = 0$, since the Fourier coefficients of $f - f_1$ are zero. This finishes the proof. □

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