Filomat 38:10 (2024), 3347–3360
https://doi.org/10.2298/FIL2410347A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectral problems of non-self-adjoint singular q -Sturm–Liouville problem with an eigenparameter in the boundary condition

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Abstract. In this paper, a non-self-adjoint (dissipative) q-Sturm–Liouville boundary-value problem in the limit-circle case with an eigenparameter in the boundary condition is investigated. The method is based on the use of the dissipative operator whose spectral analysis is sufficient for boundary value problem. A self-adjoint dilation of the dissipative operator together with its incoming and outgoing spectral representations is established and so it becomes possible to determine the scattering function of the dilation. A functional model of the dissipative operator is constructed and its characteristic function in terms of scattering function of dilation is defined. Theorems on the completeness of the system of eigenvectors and the associated vectors of the dissipative operator and the q-Sturm–Liouville boundary value problem are presented.

1. Introduction and notations

In this section, we describe some of the necessary q-notations and results (see [4, 5, 7-10, 14, 16]). Throughout the paper, q denotes a positive number such that $0 < q < 1$. For $\mu \in \mathbb{R} := (-\infty, \infty)$, a set $A \subseteq \mathbb{R}$ is called a μ-geometric set if $\mu t \in A$ for all $t \in A$. If $A \subseteq \mathbb{R}$ is a μ-geometric set, then it includes all geometric sequences $\{\mu^n t\}$ $(n = 0, 1, 2...)$, $t \in A$. Let f be a real or complex-valued function defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(t) := \frac{f(t) - f(qt)}{t - qt}, \; t \in A\backslash\{0\} \tag{1.1}$$

If $0 \in A$, the q-derivative at zero is given by

$$D_q f(0) := \lim_{n\to\infty} \frac{f(q^n t) - f(0)}{q^n t}$$

if the limit exists and it is independent of $t \in A$. Since the formulation of the extension problems requires the definition of $D_{q^{-1}}$ in a same manner to be

$$D_{q^{-1}} f(t) := \begin{cases} \frac{f(t)-f(q^{-1}t)}{t-q^{-1}t}, \; t \in A\backslash\{0\}, \\ D_q f(0), \; t = 0, \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 39A13, 33D05, 34B05, 34B24, 47B25, 47B44, 47A20, 47A40.
Keywords. q-Sturm–Liouville equation; limit-circle; spectral parameter in the boundary condition; dissipative operator; self-adjoint dilation; scattering matrix; characteristic function; completeness of the system of eigenvectors and associated vectors.
Received: 11 August 2023; Accepted: 25 September 2023
Communicated by Dragan S. Djordjević
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provided that $D_q f(0)$ exists. As a converse of the q-difference operator, Jackson's q-integration [22], is given by the following equation

$$\int_0^x f(t)d_qt := x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x),\ x \in A,$$

provided that the series is convergent, and

$$\int_a^b f(t)d_qt := \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,\ a,b \in A.$$

The q-integration for a function over $[0,\infty)$ defined by the formula ([7, 9])

$$\int_0^{\infty} f(t)\, d_qt = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

When it is required, q will be replaced by q^{-1}. The following facts, which will be frequently used, can be verified directly from the definition:

$$D_{q^{-1}} f(t) = D_q f(q^{-1}t),\ D_q^2 f(q^{-1}t) = qD_q[D_q f(q^{-1}t)] = D_{q^{-1}} D_q f(t).$$

Related to this operator there exists a non-symmetric formula for the q-differentation of a product

$$D_q[f(t)g(t)] = f(qt)D_q g(t) + g(t)D_q f(t).$$

From now on, we shall consider only the functions q-regular at zero, that is, functions satisfying

$$\lim_{n\to\infty} f(q^n t) = f(0).$$

The class of the functions being q-regular at zero contains the continuous functions. If f and g are q-regular at zero, then we have a rule of q-integration by parts given below

$$\int_0^a g(t)D_q f(t)d_qt = (fg)(a) - \int_0^a D_q g(t) f(qt)d_qt.$$

The q-difference calculus or quantum calculus was introduced at the beginning of the 19th century. Since then the subject of q-differential equations has developed and become a multidisciplinary subject ([7, 14, 16]). There exist numerous physical models including q-derivatives, q-integrals q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and q-Gamma functions, q-Euler–Maclaurin formula and their related problems (see [7, 14, 16]).

Annaby and Mansour [10] investigated a q-Sturm–Liouville eigenvalue problem and formulated a self-adjoint q-Sturm–Liouville operator in a Hilbert space. They discussed the properties of the eigenvalues and the eigenfunctions as well. Annaby et al. [8, 9] constructed the q-Titchmarsh–Weyl theory for singular q-Sturm–Liouville problems and defined q-limit-point and q-limit-circle singularities.

Since several problems of mathematical physics and mechanics result in boundary value problems with spectral parameter in the boundary conditions (see [11, 17, 18]), study of these problems receive great attention. There exist many studies about the boundary value problems with spectral parameters in the boundary conditions (see [1-3, 12, 13, 15, 17, 18, 20, 25, 26, 28]).

The principal aim of the present paper is to investigate the non-self-adjoint (dissipative) singular q-Sturm–Liouville boundary-value problem (2.7)-(2.9) with a spectral parameter in the boundary condition. For the boundary-value problem (2.7)-(2.9), it is sufficient to use the method based on the maximal dissipative operator and its spectral analysis in terms of the characteristic function. The maximal dissipative operator is constructed the spectrum of which coincides with the spectrum of the boundary-value problem (2.7)-(2.9). Then, the spectral analysis of a dilation is performed and the scattering function of a dilation is established by means of the Lax–Philips scattering theory in [23]. A functional model of the dissipative operator and specify its characteristic function in terms of scattering function of dilation are constructed. Theorems on completeness of the system of eigenvectors and associated vectors of the dissipative operator and also q-Sturm–Liouville boundary-value problem are presented.

2. Construction of the dissipative operator

We consider the following singular q-Sturm–Liouville expression

$$(Lx)(t) = \frac{1}{r(t)}[-\frac{1}{q}D_{q^{-1}}(p(t)D_q x(t)) + w(t)x(t)],\ t \in \mathbb{R}_+ := [0, \infty), \tag{2.1}$$

where p, r and w are real-valued functions defined on $\mathbb{R}_+$ and continuous at zero with $p(t) \neq 0$, $r(t) > 0$ for all $t \in \mathbb{R}_+$, and D_q is the q-difference operator given by (1.1).

We pass from the expression (2.1) to operators by introducing the Hilbert space $\mathfrak{L}^2_{r,q}(\mathbb{R}_+)$ which consists of all complex-valued functions x satisfying

$$\int_0^\infty |x(t)|^2 r(t) d_q t < +\infty$$

and with the inner product

$$(x, y) = \int_0^\infty x(t)\overline{y(t)} r(t) d_q t.$$

Let $\mathcal{D}_{\max}$ denote the linear set of all functions $x \in \mathfrak{L}^2_{r,q}(\mathbb{R}_+)$ such that x and $D_q x$ are continuous functions at 0 and $Lx \in \mathfrak{L}^2_{r,q}(\mathbb{R}_+)$. The *maximal* operator $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ is defined by the equality $\mathcal{L}_{\max} x = Lx$. For $x, y \in \mathcal{D}_{\max}$ we define the *q-Wronski determinant* (or *q-Wronskian*)

$$\mathcal{W}_q[x, y](t) = x(t)(pD_{q^{-1}}y)(t) - (pD_{q^{-1}}x)(t)y(t),\ t \in \mathbb{R}_+.$$

Given any functions $x, y \in \mathcal{D}_{\max}$, we get the following q-Green's formula (or Lagrange's identity) ([4, 5, 7-10, 21])

$$\int_0^t (Lx)(\xi))\overline{y(\xi)} d_q\xi - \int_0^t x(\xi)\overline{(Ly)(\xi)} d_q\xi = [x, y](t) - [x, y](0),\ t \in \mathbb{R}_+, \tag{2.2}$$

where $[x, y](t)$ is the Lagrange bracket defined by

$$[x, y](t) := \mathcal{W}_q[x, \overline{y}](t) = x(t)\overline{(pD_{q^{-1}}y)(t)} - (pD_{q^{-1}}x)(t)\overline{y(t)},\ t \in \mathbb{R}_+.$$

It follows directly from (2.2) that limit $[x, y](\infty) := \lim_{t\to\infty} [x, y](t)$ exists and it is finite for all $x, y \in \mathcal{D}_{\max}$. For an arbitrary function $x \in \mathcal{D}_{\max}$, $x(0)$ and $(pD_{q^{-1}}x)(0)$ can be defined as $x(0) := \lim_{t\to 0^+} x(t)$ and $(pD_{q^{-1}}x)(0) := \lim_{t\to 0^+}(pD_{q^{-1}}x)(t)$. These limits exist and they are finite (since x and $pD_{q^{-1}}x$ are continuous functions at 0). Let us consider, in $\mathfrak{L}^2_{r,q}(\mathbb{R}_+)$, the linear dense set $\mathcal{D}_{\min}$ consisting of precisely the vectors $x \in \mathcal{D}_{\max}$ with

$$x(0) = (pD_{q^{-1}})x(0) = 0,\ [x, y](\infty) = 0,\ \forall y \in \mathcal{D}_{\max} \tag{2.3}$$

Let the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$ be represented by $\mathcal{L}_{\min}$. It can be concluded from (2.3) that $\mathcal{L}_{\min}$ is symmetric. The *minimal operator* $\mathcal{L}_{\min}$ is a closed symmetric operator with deficiency indices $(2, 2)$ or $(1, 1)$, and $\mathcal{L}_{\max} = \mathcal{L}^*_{\min}$ ([4-9, 21]).

We suppose that Weyl's limit-circle case is valid for the expression L, i.e. the symmetric operator $\mathcal{L}_{\min}$ has deficiency indices $(2, 2)$ ([5, 7-9, 21]).

We mean by $\eta(t)$ and $\vartheta(t)$ the solutions (real-valued) of the equation

$$Lx = 0,\ t \in \mathbb{R}_+ \tag{2.4}$$

with the following initial conditions

$$\eta(0) = 1,\ (pD_{q^{-1}}\eta)(0) = 0,\ \vartheta(0) = 0,\ (pD_{q^{-1}}\vartheta)(0) = 1. \tag{2.5}$$

The q-Wronskian of the two solutions of (2.4) is independent of t, and the two solutions of this equation are linearly independent if and only if their q-Wronskian is nonzero. It can be derived from the conditions (2.5) and the constancy of the q-Wronskian that ([4, 5, 7-10])

$$\mathcal{W}_q[\eta,\vartheta](t)=\mathcal{W}_q[\eta,\vartheta](0)=1\ \ (t\in\mathbb{R}_+)\,. \tag{2.6}$$

As a result, η and ϑ construct a fundamental system of solutions of (2.4). Since limit-circle case is valid for L, η and ϑ belong to $\mathfrak{L}^2_{r,q}(\mathbb{R}_+)$, and furthermore $\eta,\vartheta\in\mathcal{D}_{\max}$.

Now we consider the boundary-value problem

$$(Lx)(t)=\mu x(t),\ x\in\mathcal{D}_{\max},\ t\in\mathbb{R}_+, \tag{2.7}$$

$$\delta_1x(0)-\delta_2(pD_{q^{-1}}x)(0)=\mu(\delta_1'x(0)-\delta_2'(pD_{q^{-1}}x)(0)), \tag{2.8}$$

$$[x,\eta](\infty)-\gamma[x,\vartheta](\infty)=0,\ \Im\gamma>0, \tag{2.9}$$

where μ is a complex spectral parameter, $\delta_1,\delta_2,\delta_1',\delta_2'\in\mathbb{R}:=(-\infty,\infty)$, and

$$\delta:=\begin{vmatrix}\delta_1' & \delta_1\\ \delta_2' & \delta_2\end{vmatrix}>0.$$

For convenience, we shall use the following notations:

$$G_0(x):=\delta_1x(0)-\delta_2(pD_{q^{-1}}x)(0),\ G_0'(x):=\delta_1'x(0)-\delta_2'(pD_{q^{-1}}x)(0),$$

$$E_1^+(x):=[x,\eta](\infty),\ E_2^+(x):=\mathcal{W}_q[x,\vartheta](\infty),G_+(x)=E_1^+(x)-\gamma E_2^+(x).$$

Thus for any $x,z\in\mathcal{D}_{\max}$, we have

$$\mathcal{W}_q[x,z](0)=\frac{1}{\delta}[G_0(x)G_0'(z)-G_0'(x)G_0(z)], \tag{2.10}$$

$$[x,z](t)=[x,\eta](t)[\overline{z},\vartheta](t)-[x,\vartheta](x)[\overline{z},\eta](t)\ (0\le t\le\infty), \tag{2.11}$$

$$G_0(\overline{z})=\overline{G_0(z)},E_1^+(\overline{z})=\overline{E_1^+(z)},E_2^+(\overline{z})=\overline{E_2^+(z)}.$$

Let ϕ_μ and χ_μ be the solutions of (2.7) satisfying the following conditions

$$\phi_\mu(0)=\delta_2-\delta_2'\mu,\ (pD_{q^{-1}}\phi_\mu)(0)=\delta_1-\delta_1'\mu,\ E_1^+(\chi_\mu)=\gamma,\ E_2^+(\chi_\mu)=1.$$

Equality (2.10) implies that

$$\Delta(\mu):=\mathcal{W}_q[\chi_\mu,\phi_\mu](t)=-\mathcal{W}_q[\phi_\mu,\chi_\mu](t)=-\mathcal{W}_q[\phi_\mu,\chi_\mu](t)$$

$$=-\frac{1}{\delta}[G_0(\phi_\mu)G_0'(\chi_\mu)-G_0'(\phi_\mu)G_0(\chi_\mu)]=G_0(\chi_\mu)-\mu G_0'(\chi_\mu). \tag{2.12}$$

We can see from the equality (2.11) that

$$\Delta(\mu)=-\mathcal{W}_q[\phi_\mu,\chi_\mu](t)=-\mathcal{W}_q[\phi_\mu,\chi_\mu](\infty)=-E_1^+(\phi_\mu)E_2^+(\chi_\mu)$$

$$+E_2^+(\phi_\mu)E_1^+(\chi_\mu)=-E_1^+(\phi_\mu)+\gamma E_2^+(\phi_\mu)=-G_+(\phi_\mu). \tag{2.13}$$

Spectrum of the boundary-value problem (2.7)-(2.9) coincides with the roots of the equation $\Delta(\mu)=0$. Due to the fact that Δ is analytic and not identically zero, Δ has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity.

We need to define a suitable operator in order to investigate the spectral properties of the problem (2.7)-(2.9). We denote the vector

$$F(t)=\begin{pmatrix} f_1(t)\\ f_2\end{pmatrix},$$

where $H := \mathfrak{L}^2_{r,q}(\mathbb{R}_+) \oplus \mathbb{C}$ with $f_1(.) \in \mathfrak{L}^2_{r,q}(\mathbb{R}_+)$ and $f_2 \in \mathbb{C}$. H is a Hilbert space with the inner product

$$(F, G)_H = \int_0^\infty f_1(t)\overline{g_1(t)}r(t)dt + \frac{1}{\delta} f_2 \overline{g_2},$$

where

$$F = \begin{pmatrix} f_1(t) \\ f_2 \end{pmatrix}, \; G = \begin{pmatrix} g_1(t) \\ g_2 \end{pmatrix}.$$

We consider the set given by

$$Dom(A_\gamma) = \{X = \begin{pmatrix} x_1(.) \\ x_2 \end{pmatrix} \in H : x_1 \in \mathcal{D}_{\max}, \; G_+(x_1) = 0, \; x_2 = G_0'(x_1)\},$$

Then we introduce an operator A_γ on $Dom(A_\gamma)$ as follows

$$A_\gamma X = \widetilde{L}X := \begin{pmatrix} Lx_1 \\ G_0(x_1) \end{pmatrix}.$$

We remind that a linear operator $\mathbf{T}$ (with dense domain $Dom(\mathbf{T})$) acting on a Hilbert space $\mathbf{H}$ is called *dissipative* (*accumulative*) if $\Im(\mathbf{T}f, f) \geq 0$ ($\Im(\mathbf{T}f, f) \leq 0$) for all $f \in Dom(\mathbf{T})$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative (accumulative) extension ([1-6, 19, 21]).

Theorem 2.1. *The operator A_γ is maximal dissipative in the space H.*

Proof. For $X \in Dom(A_\gamma)$, it follows from (2.10) that

$$(A_\gamma X, X)_H - (X, A_\gamma X)_H = [x_1, x_1](\infty) - [x_1, x_1](0) - E_2^+(x_1)E_1^+(\overline{x_1})$$
$$= \gamma E_2^+(x_1)E_2^+(\overline{x_1}) - \overline{\gamma}E_2^+(x_1)E_2^+(\overline{x_1}) = (\gamma - \overline{\gamma})\left|E_2^+(x_1)\right|^2$$
$$+\frac{1}{\delta}[G_0(x_1)\overline{G_0'(x_1)} - G_0'(x_1)\overline{G_0(x_1)}] = [x_1, x_1](\infty) = E_1^+(x_1)E_2^+(\overline{x_1}) \tag{2.14}$$

which implies that $\Im(A_\gamma X, X)_H = \Im\gamma\left|E_2^+(x_1)\right|^2 \geq 0$, i.e. A_γ is a dissipative operator on H.

It is not difficult to see that $(A_\gamma - \mu I)Dom(A_\gamma) = H$, $\Im\mu < 0$. Thus, A_γ is a maximal dissipative operator in H. Theorem 2.1 is proved. □

Definition 2.2. *The system of functions $x_0, x_1, ..., x_n$ is called a chain of eigenfunctions and associated functions of the boundary problem (2.7)-(2.9), corresponding to the eigenvalue μ_0, if the conditions*

$$Lx_0 = \mu_0 x_0, \; G_0(x_0) - \mu_0 G_0'(x_0) = 0, \; G_+(x_0) = 0, \tag{2.15}$$

$$Lx_k - \mu_0 x_k - x_{k-1} = 0, \; G_0(x_k) - \mu_0 G_0'(x_k) - G_0'(x_{k-1}) = 0,$$
$$G_+(x_k) = 0, \; k = 1, 2, ..., n \tag{2.16}$$

are fulfilled.

Then we obtain the following result.

Lemma 2.3. *Together with their multiplicity, the eigenvalues of the boundary-value problem (2.7)-(2.9) and the eigenvalues of the dissipative operator A_γ coincide. Each chain of eigenfunctions and associated functions $x_0, x_1, ..., x_n$ of the boundary-value problem (2.7)-(2.9), meeting the requirements of the eigenvalue μ_0, corresponds to the chain of eigenvectors and associated vectors $X_0, X_1, ..., X_n$ of the operator A_γ corresponding to the same eigenvalue μ_0. Then, the following equality*

$$X_k = \begin{pmatrix} x_k \\ G_0'(x_k) \end{pmatrix}, \; k = 0, 1, ..., n \tag{2.17}$$

holds true.

Proof. If $X_0 \in Dom(A_\gamma)$ and $A_\gamma X_0 = \mu_0 X_0$, then the equalities $Lx_0 = \mu_0 x_0$, $G_0(x_0) - \mu_0 G_0'(x_0) = 0$, $G_+(x_0) = 0$, are fulfilled, i.e. x_0 is an eigenfunction of the boundary-value problem (2.7)-(2.9). Conversely, if the conditions (2.15) are realized, then we have

$$\begin{pmatrix} x_0 \\ G_0'(x_0) \end{pmatrix} = X_0 \in Dom(A_\gamma)$$

and $A_\gamma X_0 = \mu_0 X_0$, i.e. X_0 is an eigenvector of the operator A_γ.

Moreover, if $X_0, X_1, ..., X_n$ are a chain of the eigenvectors and associated vectors of the operator A_γ corresponding to the eigenvalue μ_0, then by means of the conditions $X_k \in Dom(A_\gamma)$ $(k = 0, 1, ..., n)$ and equality $A_\gamma X_0 = \mu_0 X_0$, $A_\gamma X_k = \mu_0 X_k + X_{k-1}$, $k = 1, 2, ..., n$, we obtain the equality (2.16), where $x_0, x_1, ..., x_n$ denote the first components of the vectors $X_0, X_1, ..., X_n$. On the other side, based on the elements $x_0, x_1, ..., x_n$ related to (2.7)-(2.9), it is possible to find the vectors

$$X_k = \begin{pmatrix} x_k \\ G_0'(x_k) \end{pmatrix},$$

for which $X_k \in Dom(A_\gamma)$ $(k = 0, 1, ..., n)$ and $A_\gamma X_0 = \mu_0 X_0$, $A_\gamma X_k = \mu_0 X_k + X_{k-1}$, $k = 1, 2, ...n$. The proof of the Lemma 2.3 is completed. □

3. Self-adjoint dilation, scattering theory of dilation and functional model of dissipative operator

We deal with the Hilbert spaces $\mathfrak{L}^2(\mathbb{R}_-)$, $(\mathbb{R}_- := (-\infty, 0])$ and $\mathfrak{L}^2(\mathbb{R}_+)$ $(\mathbb{R}_- := [0, \infty))$ consisting of all functions σ_- and σ_+, respectively, such that

$$\int_{-\infty}^{0} |\sigma_-(t)|^2 \, dt < \infty, \ \int_{0}^{\infty} |\sigma_+(t)|^2 \, dt < \infty$$

with the inner product

$$(\sigma_-, \rho_-)_{\mathfrak{L}^2(\mathbb{R}_-)} = \int_{-\infty}^{0} \sigma_-(t)\overline{\rho_-(t)}dt, \ (\sigma_+, \rho_+)_{\mathfrak{L}^2(\mathbb{R}_+)} = \int_{0}^{\infty} \sigma_+(t)\overline{\rho_+(t)}dt.$$

Adding the spaces $\mathfrak{L}^2(\mathbb{R}_-)$ and $\mathfrak{L}^2(\mathbb{R}_+)$ to the Hilbert space H, we obtain an orthogonal sum Hilbert space as $\mathcal{H} = \mathfrak{L}^2(\mathbb{R}_-) \oplus H \oplus \mathfrak{L}^2(\mathbb{R}_+)$, and we call it as the *main Hilbert space of the dilation*. In $\mathcal{H}$, let us consider the set $Dom(\mathcal{S}_\gamma)$ containing all vectors $\langle \sigma_-, x, \sigma_+ \rangle$ with $\sigma_- \in W_2^1(\mathbb{R}_-)$, $\sigma_+ \in W_2^1(\mathbb{R}_+)$ ($W_2^1(\mathbb{R}_\pm)$ is the Sobolev space), $X \in H$,

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2 \end{pmatrix},$$

$x_1 \in \mathcal{D}_{\max}$, $x_2 = G_0'(x_1)$ and satisfying the conditions

$$[x_1, \eta](\infty) - \gamma[x_1, \vartheta](\infty) = \beta\sigma_-(0),$$

$$[x_1, \eta](\infty) - \overline{\gamma}[x_1, \vartheta](\infty) = \beta\sigma_+(0),$$

where $\beta^2 := 2\Im\gamma$, $\beta > 0$.

Now, we consider the operator $\mathcal{S}_\gamma$ on $Dom(\mathcal{S}_\gamma) \subset \mathcal{H}$, generated by the expression

$$\mathcal{S}\langle \sigma_-, X, \sigma_+ \rangle = \left\langle i\frac{d\sigma_-}{d\xi}, \widetilde{L}(X), i\frac{d\sigma_+}{d\zeta} \right\rangle \tag{3.1}$$

as $\mathcal{S}_\gamma F = \mathcal{S}F$, $F \in Dom(\mathcal{S}_\gamma)$. Hence we can state the next theorem.

Theorem 3.1. *The operator S_γ is self-adjoint in the space $\mathcal{H}$.*

Proof. Let us take two vectors $\Phi = \langle \sigma_-, X, \sigma_+ \rangle$, $\Psi = \langle \rho_-, Z, \rho_+ \rangle \in \mathcal{H}$. Then we see that

$$(\mathcal{S}_\gamma \Phi, \Psi)_{\mathcal{H}} - (\Phi, \mathcal{S}_\gamma \Psi)_{\mathcal{H}} = [x_1, z_1](\infty) - [x_1, z_1](0)$$
$$+\frac{1}{\delta}(G_0(x_1)\overline{G_0'(z_1)} - G_0'(x_1)\overline{G_0(z_1)}) + i\sigma_-(0)\overline{\rho_-}(0) - i\sigma_+(0)\overline{\rho_+}(0)$$
$$= [x_1, z_1](\infty) + i\sigma_-(0)\overline{\rho_-}(0) - i\sigma_+(0)\overline{\rho_+}(0) = [x_1, z_1](\infty)$$
$$-\frac{1}{i\beta^2}([x_1, \eta](\infty) - \gamma[x_1, \vartheta](\infty))(\overline{[z_1, \eta]}(\infty) - \overline{\gamma}\overline{[z_1, \vartheta]}(\infty))$$
$$+\frac{1}{i\beta^2}([x_1, \eta](\infty) - \overline{\gamma}[x_1, \vartheta](\infty))(\overline{[z_1, \eta]}(\infty) - \gamma\overline{[z_1, \vartheta]}(\infty))$$
$$= [x_1, z_1](\infty) - \frac{1}{i\beta^2}\{[x_1, \eta](\infty)\overline{[z_1, \eta]}(\infty) - \overline{\gamma}[x_1, \eta](\infty)\overline{[z_1, \vartheta]}(\infty)$$
$$-\gamma[x_1, \vartheta](\infty)\overline{[z_1, \eta]}(\infty) + \left|\gamma\right|^2 [x_1, \vartheta](\infty)\overline{[z_1, \vartheta]}(\infty)\}$$
$$+\frac{1}{i\beta^2}\{[x_1, \eta](\infty)\overline{[z_1, \eta]}(\infty) - \gamma[x_1, \eta](\infty)\overline{[z_1, \vartheta]}(\infty)$$
$$-\overline{\gamma}[x_1, \vartheta](\infty)\overline{[z_1, \eta]}(\infty) + \left|\gamma\right|^2 [x_1, \vartheta](\infty)\overline{[z_1, \vartheta]}(\infty)\} = [x_1, z_1](\infty)$$
$$-\frac{1}{i\beta^2}\{(-\overline{\gamma} + \gamma)[x_1, \eta]_\infty\overline{[z_1, \vartheta]}(\infty) + (-\gamma + \overline{\gamma})[x_1, \vartheta](\infty)\overline{[z_1, \eta]}(\infty)\}$$
$$= [x_1, z_1](\infty) - [x_1, \eta](\infty)\overline{[z_1, \vartheta]}(\infty) + [x_1, \vartheta](\infty)\overline{[z_1, \eta]}(\infty). \tag{3.2}$$

Using (2.11) and (3.2), we obtain $(\mathcal{S}_\gamma \Phi, \Psi)_{\mathcal{H}} - (\Phi, \mathcal{S}_\gamma \Psi)_{\mathcal{H}} = 0$, i.e. $\mathcal{S}_\gamma$ is a symmetric operator in $\mathcal{H}$.

In order to verify that $\mathcal{S}_\gamma$ is self-adjoint, it is sufficient to show that $\mathcal{S}_\gamma^* \subseteq \mathcal{S}_\gamma$. Let us consider the bilinear form $(\mathcal{S}_\gamma \Phi, \Psi)_{\mathcal{H}}$ on elements $\Psi = \langle \rho_-, Z, \rho_+ \rangle \in Dom(\mathcal{S}_\gamma^*)$, where $\Phi = \langle \sigma_-, 0, \sigma_+ \rangle$, $\sigma_\mp \in W_2^1(\mathbb{R}_\mp)$, $\sigma_\mp(0) = 0$. If we use integration by parts, we get $\mathcal{S}_\gamma^* \Psi = \langle i\frac{d\rho_-}{d\xi}, Z^*, i\frac{d\rho_+}{d\zeta} \rangle$, where $\rho_\mp \in W_2^1(\mathbb{R}_\mp)$, $Z^* \in H$. In a similar manner, if $\Phi = \langle 0, X, 0 \rangle \in Dom(\mathcal{S}_\gamma)$, then using integration by parts in $(\mathcal{S}_\gamma \Phi, \Psi)_{\mathcal{H}}$, we find that

$$\mathcal{S}_\gamma^* \Psi = \mathcal{S}_\gamma^* \langle \rho_-, Z, \rho_+ \rangle$$
$$= \left\langle i\frac{d\rho_-}{d\xi}, \widetilde{L}(Z), i\frac{d\rho_+}{d\zeta} \right\rangle,\ z_1 \in \mathcal{D}_{\max},\ z_2 = G_0'(z_1). \tag{3.3}$$

Consequently, it follows from (3.3) that $(\mathcal{S}\Phi, \Psi)_{\mathcal{H}} = (\Phi, \mathcal{S}\Psi)_{\mathcal{H}}$, $\forall \Phi \in Dom(\mathcal{S}_\gamma)$, where the operator $\mathcal{S}$ is given by (3.1). Hence, the sum of the integrated terms in the bilinear form $(\mathcal{S}\Phi, \Psi)_{\mathcal{H}}$ must be zero:

$$[x_1, z_1](\infty) - [x_1, z_1](0) + \frac{1}{\delta}[G_0(x_1)\overline{G_0'(z_1)} - G_0'(x_1)\overline{G_0(z_1)}]$$
$$+i\sigma_-(0)\overline{\rho_-(0)} - i\sigma_+(0)\overline{\rho_+(0)} = 0. \tag{3.4}$$

Using the equation (2.10), we can see that

$$[x_1, z_1](\infty) + i\sigma_-(0)\overline{\rho_-(0)} - i\sigma_+(0)\overline{\rho_+(0)} = 0. \tag{3.5}$$

Moreover, boundary conditions for $\mathcal{S}_\gamma$ imply that

$$[x_1,\eta](\infty) = \beta\sigma_-(0) + \frac{i\gamma}{\beta}(\sigma_-(0) - \sigma_+(0)),$$

$$[x_1,\vartheta](\infty) = \frac{i}{\beta}(\sigma_-(0) - \sigma_+(0)).$$

Then (2.11) and (3.5) lead to

$$[\beta\sigma_-(0) + \frac{i\gamma}{\beta}(\sigma_-(0) - \sigma_+(0))]\overline{[z_1,\vartheta]}(\infty)$$

$$-\frac{i}{\beta}(\sigma_-(0) - \sigma_+(0))\overline{[z_1,\eta]}(\infty)\sigma = i\sigma_+(0)\overline{\rho_+(0)} - i\sigma_-(0)\overline{\rho_-(0)}. \tag{3.6}$$

If we compare the coefficients of $\sigma_-(0)$ in (3.6), we have

$$\frac{i\beta^2-\gamma}{\beta}\overline{[z_1,\vartheta]}(\infty) + \frac{1}{\beta}\overline{[z_1,\eta]}(\infty) = \overline{\rho_-(0)}$$

or

$$[z_1,\eta](\infty) - \gamma[z_1,\vartheta](\infty) = \beta\rho_-(0). \tag{3.7}$$

Similarly, when the coefficients of $\sigma_+(0)$ in (3.6) are compared, it is seen that

$$[z_1,\eta](\infty) - \overline{\gamma}[z_1,\vartheta](\infty) = \beta\rho_+(0). \tag{3.8}$$

As a result, conditions (3.7) and (3.8) give us that $Dom(\mathcal{S}_\gamma^*) \subseteq Dom(\mathcal{S}_\gamma)$, and thus $\mathcal{S}_\gamma = \mathcal{S}_\gamma^*$, that is, the theorem is proved. □

The self-adjoint operator $\mathcal{S}_\gamma$ generates the unitary group $\mathfrak{X}_\gamma(s) = \exp(i\mathcal{S}_\gamma s)$ $(s \in \mathbb{R})$ on $\mathcal{H}$. Let $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ denote the mappings acting according to the formulas $P : \langle\sigma_-, H, \sigma_+\rangle \to H$ and $P_1 : H \to \langle 0, H, 0\rangle$. Define $\mathfrak{Z}_\gamma(s) := P\mathfrak{X}_\gamma(s)P_1, s \geq 0$. The family $\left\{\mathfrak{Z}_\gamma(s)\right\}$ $(s \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contractions on H. Let us denote by B_γ the generator of this semigroup: $B_\gamma X = \lim_{s\to+0}(is)^{-1}(\mathfrak{Z}_\gamma(s)X - X)$. The domain of B_γ is composed of all vectors for which the limit exists. The operator B_γ is dissipative. The operator $\mathcal{S}_\gamma$ is defined as the *self-adjoint dilation* of B_γ ([1-5, 24, 27]).

Then the next theorem can be stated.

Theorem 3.2. *The operator S_γ is a self-adjoint dilation of the dissipative operator A_γ.*

Proof. We shall show that $B_\gamma = A_\gamma$, and hence it will be obtained that $\mathcal{S}_\gamma$ is a self-adjoint dilation of A_γ. To do this, we start with verifying the equality

$$P(\mathcal{S}_\gamma - \mu I)^{-1}P_1 X = (A_\gamma - \mu I)^{-1}X,\ X \in H,\ \Im\mu < 0. \tag{3.9}$$

For this purpose, we set $(\mathcal{S}_\gamma - \mu I)^{-1}P_1 X = \Psi = \langle\rho_-, Z, \rho_+\rangle$. Then $(\mathcal{S}_\gamma - \mu I)\Psi = P_1 X$, and thus, $\widetilde{L}(Z) - \mu Z = X$, $\rho_-(\xi) = \rho_-(0)e^{-i\mu\xi}$ and $\rho_+(\zeta) = \rho_+(0)e^{-i\mu\zeta}$. Since $\Psi \in Dom(\mathcal{S}_\gamma)$, then $\rho_- \in \mathfrak{L}^2(\mathbb{R}_-)$; it leads to $\rho_-(0) = 0$, and in conclusion, Z satisfies the boundary condition $[z_1,\eta](\infty) - \gamma[z_1,\vartheta](\infty) = 0$. Thus, $Z \in Dom(A_\gamma)$, and since a point μ with $\Im\mu < 0$ can not be an eigenvalue of a dissipative operator, it means that $\rho_+(0)$ is obtained from the formula $\rho_+(0) = \beta^{-1}\{[z_1,\eta](\infty) - \overline{\gamma}[z_1,\vartheta](\infty)\}$. Therefore,

$$(\mathcal{S}_\gamma - \mu I)^{-1}P_1 X = \left\langle 0, (A_\gamma - \mu I)^{-1}X, \beta^{-1}([z_1,\eta](\infty) - \overline{\gamma}[z_1,\vartheta](\infty))\right\rangle$$

for $X \in H$ and $\Im\mu < 0$. We find by using the mapping P to (3.9) that

$$(A_\gamma - \mu I)^{-1} = P(\mathcal{S}_\gamma - \mu I)^{-1}P_1 = -iP\int_0^\infty \mathfrak{X}_\gamma(s)e^{-i\mu s}ds P_1$$

$$= -i\int_0^\infty \mathfrak{Z}_\gamma(s)e^{-i\mu s}ds = (B_\gamma - \mu I)^{-1}, \Im\mu < 0,$$

which implies that $A_\gamma = B_\gamma$, completing the proof. □

The unitary group $\mathfrak{X}_\gamma(s) = \exp[i\mathcal{S}_\gamma s]$ ($s \in \mathbb{R}$) has a crucial meaning as we can apply to it the Lax–Phillips scheme [23]. We consider the subspaces $\mathcal{D}_- = \left\langle \mathfrak{L}^2(\mathbb{R}_-), 0, 0\right\rangle$ and $\mathcal{D}_+ = \left\langle 0, 0, \mathfrak{L}^2(\mathbb{R}_+)\right\rangle$ in $\mathcal{H}$. Then $\mathcal{D}_-$ and $\mathcal{D}_+$ have the following features:

(1) $\mathfrak{X}_\gamma(s)\mathcal{D}_- \subset \mathcal{D}_-$, $s \le 0$ and $\mathfrak{X}_\gamma(s)\mathcal{D}_+ \subset \mathcal{D}_+$, $s \ge 0$;

(2) $\bigcap_{s\le 0}\mathfrak{X}_\gamma(s)\mathcal{D}_- = \bigcap_{s\ge 0}\mathfrak{X}_\gamma(s)\mathcal{D}_+ = \{0\}$;

(3) $\overline{\bigcup_{s\ge 0}\mathfrak{X}_\gamma(s)\mathcal{D}_-} = \overline{\bigcup_{s\le 0}\mathfrak{X}_\gamma(s)\mathcal{D}_+} = \mathcal{H}$;

(4) $\mathcal{D}_- \perp \mathcal{D}_+$.

Property **(4)** is clear. We handle with the proof (**1**) for $\mathcal{D}_+$ (a similar proof can be given for $\mathcal{D}_-$). Consider the operator for the vector $\Phi = \langle 0, 0, \sigma_+\rangle \in \mathcal{D}_+$

$$R_\mu\Phi = \left\langle 0, 0, -ie^{-i\mu\xi}\int_0^\xi e^{i\mu s}\sigma_+(s)ds\right\rangle.$$

which is denoted by $R_\mu\Phi = (\mathcal{S}_\gamma - \mu I)^{-1}\Phi$. A direct computation indicates that $R_\mu\Phi \in \mathcal{D}_+$. Hence we have for $\Psi \perp \mathcal{D}_+$ that

$$(R_\mu\Phi, \Psi)_\mathcal{H} = -i\int_0^\infty e^{-i\mu s}(\mathfrak{X}_\gamma(s)\Phi, \Psi)_\mathcal{H}ds = 0,\ \Im\mu < 0,$$

and in turn $(\mathfrak{X}_\gamma(s)\Phi, \Psi)_\mathcal{H} = 0$ for all $s \ge 0$. This leads to $\mathfrak{X}_\gamma(s)\mathcal{D}_+ \subset \mathcal{D}_+$, for $s \ge 0$, and thus property (**1**) is proved.

We know that the generator of the semigroup $\mathcal{V}(s)$ of the one-sided shift in the space $\mathfrak{L}^2(\mathbb{R}_+)$ is the differential operator $i\frac{d}{d\xi}$ satisfying the boundary condition $\sigma(0) = 0$. We define the semigroup of isometries $\mathfrak{X}^+_\gamma(s) := P^+\mathfrak{X}_\gamma(s)P^+_1$, $s \ge 0$, where $P^+ : \mathcal{H} \to \mathfrak{L}^2(\mathbb{R}_+)$ and $P^+_1 : \mathfrak{L}^2(\mathbb{R}_+) \to \mathcal{D}_+$ ($P^+ : \langle\sigma_-, X, \sigma_+\rangle \to \sigma_+$, $P^+_1 : \sigma \to \langle 0, 0, \sigma\rangle$). Besides, the generator S of the semigroup of isometries $\mathfrak{X}^+_\gamma(s), s \ge 0$, is the operator

$$S\sigma = P^+\mathcal{S}_\gamma P^+_1\sigma = P^+\mathcal{S}_\gamma\langle 0, 0, \sigma\rangle = P^+\left\langle 0, 0, i\frac{d\sigma}{d\xi}\right\rangle = i\frac{d\sigma}{d\xi},$$

where $\sigma \in W^1_2(\mathbb{R}_+)$ and $\sigma(0) = 0$. Since a semigroup is uniquely determined by its generator, it is seen that $\mathfrak{X}^+_\gamma(s) = \mathcal{V}(s)$. Consequently, the following equality

$$\bigcap_{s\ge 0}\mathfrak{X}^+_\gamma(s)\mathcal{D}_+ = \left\langle 0, 0, \bigcap_{s\ge 0}\mathcal{V}(s)\mathfrak{L}^2(\mathbb{R}_+)\right\rangle = \{0\},$$

shows that property (**2**) is fulfilled.

According to the scheme of the Lax–Phillips scattering theory ([23]), the scattering function is determined in terms of the theory of spectral representations. Now let us proceed to their construction. During this construction, property (**3**) of the incoming and outgoing subspaces will be proven.

Recall that a linear operator **B** (with domain $\mathcal{D}(\mathbf{B})$) acting in a Hilbert space **H** is called *completely non-self-adjoint* (or *pure*) if the invariant subspace $\mathbf{M} \subseteq \mathcal{D}(\mathbf{B})$ ($\mathbf{M} \neq \{0\}$) of the operator **B** whose restriction to **M** is self-adjoint, does not exist.

Then we have the next conclusion.

Lemma 3.3. *The operator A_γ is completely non-self-adjoint (pure).*

Proof. Assume on the contrary that $H' \subset H$ is a nontrivial subspace in which A_γ induces a self-adjoint operator A'_γ with domain $Dom(A'_\gamma) = H' \cap Dom(A_\gamma)$. If $Z \in Dom(A'_\gamma)$, then $Z \in Dom(A'^*_\gamma)$ and thus

$$0 = (A'_\gamma Z, Z)_H - (Z, A'_\gamma Z)_H = [z_1, z_1](\infty) - i[z_1, z_1](0)$$

$$+\frac{1}{\delta}[G_0(z_1)\overline{G_0'(z_1)} - G_0'(z_1)\overline{G_0(z_1)}]$$
$$= 2i\Im\gamma\left|E_2^+(z_1)\right|^2 = i\beta^2\left|\mathcal{W}_q[z_1,\vartheta](\infty)\right|^2.$$

This means that we have $[x_1,\vartheta](\infty) = 0$ for the eigenvectors $X(t,\mu)$ of the operator A_γ' that lie in H' and are eigenvectors of A_γ. Using the equality $[x_1,\eta](\infty) - \gamma[x_1,\vartheta](\infty) = 0$ we find $[x_1,\eta](\infty) = 0$, which implies that $X(t,\mu) = 0$. Hence, it follows from the theorem on expansion in eigenvectors of the self-adjoint operator A_γ' that $H' = \{0\}$. This contradiction completes the proof. □

Now consider the following spaces

$$\mathcal{H}_- = \overline{\bigcup_{s\geq 0}\mathfrak{X}_\gamma(s)\mathcal{D}_-},\ \mathcal{H}_+ = \overline{\bigcup_{s\leq 0}\mathfrak{X}_\gamma(s)\mathcal{D}_+}.$$

Lemma 3.4. $\mathcal{H}_- + \mathcal{H}_+ = \mathcal{H}$.

Proof. The subspace $\mathcal{H}' = \mathcal{H}\ominus(\mathcal{H}_- + \mathcal{H}_+)$ is invariant relative to the group $\left\{\mathfrak{X}_\gamma(s)\right\}$. In order to show this, it suffices to consider the property (**1**) of the subspace $\mathcal{D}_+$. In addition, it can be regarded that $\mathcal{H}'$ is in the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Then, if the subspace $\mathcal{H}'$ (and hence also H') were nontrivial, then the unitary group $\left\{\mathfrak{X}_\gamma'(s)\right\}$, restricted to this subspace, would be a unitary part of the group $\left\{\mathfrak{X}_\gamma(s)\right\}$, and consequently, the restriction A_γ' of A_γ to H' would be a self-adjoint operator in H'. Purity of the operator A_γ gives us that $H' = \{0\}$. The lemma is proved. □

Let $\theta_\mu(t)$ and $\phi_\mu(t)$ be solutions of the equation (2.7) satisfying the conditions

$$\theta_\mu(0) = \frac{\delta_2'}{\delta},\ (pD_{q^{-1}}\theta_\mu)(0) = \frac{\delta_1'}{\delta},$$
$$\phi_\mu(0) = \delta_2 - \delta_2'\mu,\ (pD_{q^{-1}}\phi_\mu)(0) = \delta_1 - \delta_1'\mu.$$

Let us define the following notations:

$$n(\mu) := \frac{[\theta_\mu,\vartheta](\infty)}{[\phi_\mu,\vartheta](\infty)},\ \upsilon(\mu) := -\frac{[\phi_\mu,\eta](\infty)}{[\phi_\mu,\vartheta](\infty)},\ \Phi_\mu(t) := \begin{pmatrix}\phi_\mu(t)\\ \delta\end{pmatrix}, \tag{3.10}$$

$$\Theta_\gamma(\mu) := \frac{\upsilon(\mu)+\gamma}{\upsilon(\mu)+\overline{\gamma}}. \tag{3.11}$$

It follows from (3.10) that $\upsilon(\mu)$ is a meromorphic function on the complex plane $\mathbb{C}$ having a countable number of poles on the real axis. Note that $\upsilon(\mu)$ has the property: $\Im\mu\Im\upsilon(\mu) < 0$, $\Im\mu \neq 0$ and $\overline{\upsilon(\mu)} = \upsilon(\overline{\mu})$ for all $\mu\in\mathbb{C}$, except the real poles of $\upsilon(\mu)$.

Consider the vector-valued function

$$\mathcal{V}_\mu^-(t,\xi,\zeta)$$
$$= \left\langle e^{-i\mu\xi}, \delta n(\mu)\left\{(\upsilon(\mu)+\gamma)[\theta_\mu,\vartheta](\infty)\right\}^{-1}\Phi_\mu(t), \overline{\Theta}_\gamma(\mu)e^{-i\mu\zeta}\right\rangle, \tag{3.12}$$

where $t,\zeta\in\mathbb{R}_+$, $\xi\in\mathbb{R}_-$. Using the vector $\Phi = \langle\sigma_-, X, \sigma_+\rangle$, we consider the transformation $\Upsilon_- : \Phi\to\widetilde{\Phi}_-(\mu)$ by

$$(\Upsilon_-\Phi)(\mu) := \widetilde{\Phi}_-(\mu) := \frac{1}{\sqrt{2\pi}}(\Phi,\mathcal{V}_\mu^-)_{\mathcal{H}}$$

on the vector $\Phi = \langle\sigma_-, X, \sigma_+\rangle$, where σ_-, σ_+ and x_1 are smooth, compactly supported functions.

Lemma 3.5. *The transformation Υ_- isometrically maps $\mathcal{H}_-$ onto $\mathfrak{L}^2(\mathbb{R})$. For all vectors $\Phi, \Psi \in \mathcal{H}_-$ the Parseval equality and the inversion formula are satisfied:*

$$(\Phi, \Psi)_{\mathcal{H}} = (\widetilde{\Phi}_-, \widetilde{\Psi}_-)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \widetilde{\Phi}_-(\mu)\overline{\widetilde{\Psi}_-(\mu)}d\mu,$$

$$\Phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{\Phi}_-(\mu)\mathcal{V}_\mu^- d\mu,$$

where $\widetilde{\Phi}_-(\mu) = (\Upsilon_-\Phi)(\mu)$ *and* $\widetilde{\Psi}_-(\mu) = (\Upsilon_-\Psi)(\mu)$.

Proof. For the vectors $\Phi, \Psi \in \mathcal{D}_-, \Phi = \langle \sigma_-, 0, 0 \rangle, \Psi = \langle \rho_-, 0, 0 \rangle$, we have

$$\widetilde{\Phi}_-(\mu) = \frac{1}{\sqrt{2\pi}}(\Phi, \mathcal{V}_\mu^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \sigma_-(\xi)e^{i\mu\xi}d\xi \in H_-^2,$$

where $H_\pm^2$ is the Hardy class consisting of the functions in $\mathfrak{L}^2(\mathbb{R})$ which are analytically extendable to the upper and lower half-planes, respectively, and using Parseval equality for Fourier integrals

$$(\Phi, \Psi)_{\mathcal{H}} = \int_{-\infty}^{\infty} \sigma_-(\xi)\overline{\rho_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{\Phi}_-(\mu)\overline{\widetilde{\Psi}_-(\mu)}d\mu = (\Upsilon_-\Phi, \Upsilon_-\Psi)_{\mathfrak{L}^2}.$$

Let us extend the Parseval equality to the whole of $\mathcal{H}_-$. For that purpose, let $\mathcal{H}_-$ be a dense set in $\mathcal{H}_-'$ including all smooth, compactly supported functions in $\mathcal{D}_-$ with $\Phi \in \mathcal{H}'$ if $\Phi = \mathfrak{X}_\gamma(s)\Phi_0$, $\Phi_0 = \langle \sigma_-, 0, 0 \rangle$, $\sigma_- \in C_0^\infty(\mathbb{R}_-)$, where $s = s_\Phi$ is a non-negative number (depending on Φ). For $\Phi, \Psi \in \mathcal{H}_-'$ we see that $\mathfrak{X}_\gamma(-s)\Phi, \mathfrak{X}_\gamma(-s)\Psi \in \mathcal{D}_-$, (for $s > s_\Phi$ and $s > s_\Psi$) and, in addition, the first components of these vectors are elements of $C_0^\infty(\mathbb{R}_-)$. Since operators $\mathfrak{X}_\gamma(s)$ ($s \in \mathbb{R}$) are unitary, the equality

$$\Upsilon_-\mathfrak{X}_\gamma(s)\Phi = (\mathfrak{X}_\gamma(s)\Phi, U_\mu^-)_{\mathcal{H}} = e^{i\mu s}(\Phi, U_\mu^-)_{\mathcal{H}} = e^{i\mu s}\Upsilon_-\Phi$$

implies that

$$\begin{aligned}(\Phi, \Psi)_{\mathcal{H}} &= (\mathfrak{X}_\gamma(-s)\Phi, \mathfrak{X}_\gamma(-s)\Psi)_{\mathcal{H}} = \left(\Upsilon_-\mathfrak{X}_\gamma(-s)\Phi, \Upsilon_-\mathfrak{X}_\gamma(-s)\Psi\right)_{\mathfrak{L}^2} \\ &= \left(e^{-i\mu s}\Upsilon_-\Phi, e^{-i\mu s}\Upsilon_-\Psi\right)_{\mathfrak{L}^2} = (\tilde{f}, \widetilde{\Psi})_{\mathfrak{L}^2}. \end{aligned} \tag{3.13}$$

If we take the closure in (3.13), we find the Parseval equality for the space $\mathcal{H}_-$. We notice that the inversion formula is obtained from the Parseval equality if all integrals in it are interpreted as limits in the mean of integrals over finite intervals. Accordingly,

$$\Upsilon_-\mathcal{H}_- = \overline{\bigcup_{s\geq 0} \Upsilon_-\mathfrak{X}_\gamma(s)\mathcal{D}_-} = \overline{\bigcup_{s\geq 0} e^{-i\mu s}H_-^2} = \mathfrak{L}^2(\mathbb{R}),$$

i.e. Υ_- maps $\mathcal{H}_-$ onto the whole $\mathfrak{L}^2(\mathbb{R})$. The lemma is proved. □

Let us set

$$\mathcal{V}_\mu^+(t, \xi, \zeta) = \left\langle \Theta_\gamma(\mu)e^{-i\mu\xi}, \delta n(\mu)\left\{(v(\mu) + \overline{\gamma})[\theta_\mu, \vartheta](\infty)\right\}\Phi_\mu(t), e^{-i\mu\zeta} \right\rangle, \tag{3.14}$$

where $t, \zeta \in \mathbb{R}_+, \xi \in \mathbb{R}_-$ and consider the transformation $\Upsilon_+ : \Phi \to \widetilde{\Phi}_-(\mu)$ by

$$(\Upsilon_+\Phi)(\mu) := \widetilde{\Phi}_-(\mu) := \frac{1}{\sqrt{2\pi}}(\Phi, \mathcal{V}_\mu^+)_{\mathcal{H}}$$

on the vector $\Phi = \langle \sigma_-, X, \sigma_+ \rangle$, where σ_-, σ_+ and x_1 are smooth, compactly supported functions.

The proof of the next results is similar to that of Lemma 3.5.

Lemma 3.6. *The transformation Υ_+ isometrically maps $\mathcal{H}_+$ onto $\mathfrak{L}^2(\mathbb{R})$, and for all vectors $\Phi, \Psi \in \mathcal{H}_+$, the Parseval equality and the inversion formula hold:*

$$(\Phi, \Psi)_{\mathcal{H}} = (\widetilde{\Phi}_+, \widetilde{\Psi}_+)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \widetilde{\Phi}_+(\mu)\overline{\widetilde{\Psi}_+(\mu)}d\mu,$$

$$\Phi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\widetilde{\Phi}_+(\mu)\mathcal{V}_\mu^+ d\mu,$$

where $\widetilde{\Phi}_+(\mu) = (\Upsilon_+\Phi)(\mu)$ and $\widetilde{\Psi}_+(\mu) = (\Upsilon_+\Psi)(\mu)$.

It follows from (3.11) that the function Θ_γ satisfies $\left|\Theta_\gamma(\mu)\right| = 1$ for $\mu \in \mathbb{R}$. Hence, the formulas for the vectors $\mathcal{V}_\mu^-$ and $\mathcal{V}_\mu^+$ give us that

$$\mathcal{V}_\mu^- = \overline{\Theta}_\gamma(\mu)\mathcal{V}_\mu^+ \, (\mu \in \mathbb{R}). \tag{3.15}$$

Lemmas 3.5 and 3.6 lead to $\mathcal{H}_- = \mathcal{H}_+$. Together with Lemma 3.4, this means that $\mathcal{H} = \mathcal{H}_- = \mathcal{H}_+$, and thus property (**3**) above has been established for the incoming and outgoing subspaces.

Therefore, Υ_- isometrically maps $\mathcal{H}_-$ onto $\mathfrak{L}^2(\mathbb{R})$ with the subspace $\mathcal{D}_-$ mapped onto H_-^2, and the operators $\mathfrak{X}_\gamma(s)$ are transformed by the operators of multiplication by $e^{i\mu s}$. We see that Υ_- (Υ_+) is the incoming (outgoing) spectral representation for the group $\left\{\mathfrak{X}_\gamma(s)\right\}$. By means of (3.15), we can pass from the Υ_+-representation of the vector $\Phi \in \mathcal{H}$ to its Υ_--representation by multiplication of the function $\Theta_\gamma(\mu) : \widetilde{\Phi}_-(\mu) = \Theta_\gamma(\mu)\widetilde{\Phi}_+(\mu)$. Based on [23], (Chapters II, III), the *scattering function* of the group $\left\{\mathfrak{X}_\gamma(s)\right\}$ with respect to the subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$, is the coefficient by which the Υ_--representation of a vector $\Phi \in \mathcal{H}$ must be multiplied to get the corresponding Υ_+-representation: $\widetilde{\Phi}_+(\mu) = \overline{\Theta}_\gamma(\mu)\widetilde{\Phi}_-(\mu)$. As a result, the following theorem has been proved.

Theorem 3.7. *The function $\overline{\Theta}_\gamma$ is the scattering function of the group $\left\{\mathfrak{X}_\gamma(s)\right\}$ (of the self-adjoint operator S_γ).*

Let Θ be an arbitrary nonconstant inner function [24] defined on the upper half-plane $\mathbb{C}_+$ (we recall that a function Θ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $\left|\Theta(\mu)\right| \leq 1$ for $\mu \in \mathbb{C}_+$, and $\left|\Theta(\mu)\right| = 1$ for almost all $\mu \in \mathbb{R}$). We know that the subspace $\mathcal{M} = \mathcal{H}_+^2 \ominus \Theta\mathcal{H}_+^2$ is nontrivial. We consider the semigroup of the operators $\mathcal{X}(s)$ ($s \geq 0$) acting in $\mathcal{M}$ with respect to the formula $\mathcal{X}(s)u = \mathcal{P}\left[e^{i\mu s}u\right]$, $u := u(\mu) \in \mathcal{M}$, where $\mathcal{P}$ denotes the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{M}$. The generator of the semigroup $\{\mathcal{X}(s)\}$ is defined by $\mathcal{B} : \mathcal{B}u = \lim_{s\to+0}[(is)^{-1}(\mathcal{X}(s)u - u)]$, which is a dissipative operator acting in $\mathcal{M}$ with domain $Dom(\mathcal{B})$ having all functions $u \in \mathcal{M}$ for which the limit above exists. The operator $\mathcal{B}$ is known as *model dissipative operator* in the literature. Remark that this model dissipative operator belongs to Lax and Phillips [23]. A more general model dissipative operator has been constructed by Sz.-Nagy and Foiaş [24]. The basic assertion is that $\Theta(\mu)$ is the *characteristic function* of the operator $\mathcal{B}$.

Let $\mathcal{N} = \langle 0, H, 0\rangle$, and thus $\mathcal{H} = \mathcal{D}_- \oplus \mathcal{N} \oplus \mathcal{D}_+$. We obtain from the explicit form of the unitary transformation Υ_- that under the mapping Υ_-

$$\mathcal{H} \to \mathfrak{L}^2(\mathbb{R}),\ \Phi \to \widetilde{\Phi}_-(\mu) = (\Upsilon_-\Phi)(\mu),\ \mathcal{D}_- \to H_-^2,\ \mathcal{D}_+ \to \Theta_\gamma H_+^2,$$

$$\mathcal{N} \to H_+^2 \ominus \Theta_\gamma H_+^2,\ \mathfrak{X}_\gamma(s)\Phi \to (\Upsilon_-\mathfrak{X}_\gamma(s)\Upsilon_-^{-1}\widetilde{\Phi}_-)(\mu) = e^{i\mu s}\widetilde{\Phi}_-(\mu). \tag{3.16}$$

The formulas given by (3.16) imply that our operator A_γ is unitarily equivalent to the model dissipative operator with the characteristic function $\Theta_\gamma(\mu)$. Due to the fact that the characteristic functions of unitarily equivalent dissipative operators coincide ([1-5, 24, 27]), we have proved the following result.

Theorem 3.8. *The characteristic function of the dissipative operator A_γ coincides with the function Θ_γ given in* (3.11).

4. Completeness theorems of the dissipative operator A_γ and the boundary-value problem (2.7)-(2.9)

Characteristic function is very useful to answer the question that whether all eigenfunctions and associated functions of a maximal dissipative operator A_γ span the whole space or not. We can perform this analysis by ensuring that the singular factor $s(\mu)$ in the factorization $\Theta_\gamma(\mu) = s(\mu)B(\mu)$ ($B(\mu)$ is a Blaschke product) is absent ([1-5, 24, 27]).

Theorem 4.1. *For all values of γ with $\Im\gamma > 0$, except possibly for a single value $\gamma = \gamma_0$, the characteristic function Θ_γ of the dissipative operator A_γ is a Blaschke product. The spectrum of A_γ is purely discrete and lies in the open upper half-plane. The operator A_γ ($\gamma \neq \gamma_0$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of eigenvectors and associated vectors of operator A_γ ($\gamma \neq \gamma_0$) is complete in the space H.*

Proof. Using (3.11), it is easy to see that Θ_γ is an inner function in the upper half-plane and it is meromorphic in the whole μ-plane. We have the factorization

$$\Theta_\gamma(\mu) = e^{i\mu b}B_\gamma(\mu), \tag{4.1}$$

where $B_\gamma(\mu)$ is the Blaschke product and $b = b(\gamma) \geq 0$. Therefore we obtain from (4.1) that

$$\left|\Theta_\gamma(\mu)\right| = \left|e^{i\mu b}\right|\left|B_\gamma(\mu)\right| \leq e^{-b(\gamma)\Im\mu}, \Im\mu \geq 0. \tag{4.2}$$

On the other hand, if we express $v(\mu)$ in terms of $\Theta_\gamma(\mu)$ we get from (4.1) that

$$v(\mu) = \frac{\gamma - \overline{\gamma}\Theta_\gamma(\mu)}{\Theta_\gamma(\mu) - 1}. \tag{4.3}$$

If $b(\gamma) > 0$ for a given value γ ($\Im\gamma > 0$), then (4.1) gives us that $\lim_{s\to+\infty}\Theta_\gamma(is) = 0$, and then (4.3) leads to $\lim_{s\to+\infty} v(is) = -\gamma$. $v(\mu)$ can be nonzero at not more then a single point $\gamma = \gamma_0$ (and, further, $\gamma_0 = -\lim_{s\to+\infty} v(is)$) as $v(\mu)$ is independent of γ. Therefore the proof is completed. □

We have by Lemma 3.3 that the eigenvalues of the boundary-value problem (2.7)-(2.9) and the eigenvalues of the operator A_γ coincide, including their multiplicity; furthermore, for the eigenfunctions and associated functions of the boundary problems (2.7)-(2.9), the formula (2.14) is satisfied, then Theorem 4.1 can be interpreted as follows.

Theorem 4.2. *The spectrum of boundary-value problem (2.7)-(2.9) is purely discrete and belongs to the open upper half-plane. For all the values of γ with $\Im\gamma > 0$, except possibly for a single value $\gamma = \gamma_0$, the boundary-value problem (2.7)-(2.9) ($\gamma \neq \gamma_0$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of eigenfunctions and associated functions of this problem ($\gamma \neq \gamma_0$) is complete in the space $\mathfrak{L}^2_{r,q}(\mathbb{R}_+)$.*

Since a linear operator $\mathfrak{T}$ acting in the Hilbert space $\mathfrak{H}$ is maximal accumulative if and only if $-\mathfrak{T}$ is maximal dissipative, all results concerning maximal dissipative operators can be immediately stated for maximal accumulative operators. Then the Theorem 4.2 yields the following result.

Corollary 4.3. *For $\Im\gamma < 0$ the spectrum of the boundary-value problem (2.8)-(2.10) is purely discrete and belongs to the open lower half-plane. For all values of γ with $\Im\gamma < 0$, with the possible exception of a single value $\gamma = \gamma_1$, the boundary-value problem (2.8)-(2.10) ($\gamma \neq \gamma_1$) has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. The system of eigenvectors and associated vectors of this problem ($\gamma \neq \gamma_1$) is complete in the space $\mathfrak{L}^2_{r,q}(\mathbb{R}_+)$.*

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