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EVALUATION OF SOME PARAMETERS OF PARTIAL DIFFERENTIAL EQUATIONS

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Main parameters in dynamics of populations, epidemics and in demography are specific rates of birth and mortality. In our report we consider the following linear model of the "evolution equation":

$$\begin{cases} \frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+\mu(a,t,P(t),\theta)u=0 & a,t>0, \\ u(0,t)=\int\limits_0^A \beta(a,t,P(t))u(a,t)\,da & t,A\geq 0, \\ u(a,0)=\varphi(a) & a\geq 0, \end{cases} \tag{1}$$

$$P(t)=\int\limits_0^A u(a,t)\,da \quad t\geq 0,$$

where a is the age, t– the time, $u(a,t)$– the quantity of individuals of the age a at the time t, $\mu(a)$– the mortality rate, $\beta(a)$– the birth rate, $\varphi(a)$ is the age distribution of the population at the initial time, $P(t)$ is the size of the population at the time t, A is the limit age such that no individual exceeds it, θ is the vector of the unknown parameters. One is able to observe a vector $y(t,a)$ which is a variant of the state parameters $u(a,t)$ influenced by some noise.

$$y(t,a)=u(a,t)+\zeta(a,t). \tag{2}$$

where $\zeta(a,t)$ is the vector of error measurements.

The problem is to find a vector of parameters $\theta^*(t) \subset \Theta \subset R^n$, such that $y(a,t,\theta)$ (the solution of problem (1),(2)) does not differ essentially from the given $v(a)$ - the desireable function of the population distribution. In many cases due to the presence of $\zeta(t)$ it is impossible to find the exact value of $\theta(t)$.

We shall evaluate $\theta(t)$ by the criterion function

$$J = \max_{t,a}\{\varphi[(y(\cdot),\theta,\zeta,t)]\}$$

where $\varphi(\cdot)$ - is a nonnegative convex function such that

$$\varphi(0) = 0, \varphi(y) \to \infty, as \;\; y \to \infty.$$

The identification problem is to define a vector $\theta \in \Theta$ such that the functional

$$\max_{t\in[t_0,t_1]} \max_{a\in[a_0,a_1]} |v(a,t) - y(a,t,\theta)| = \Phi(\theta) \tag{3}$$

attains its minimum. Here $y(a,t,\theta)$ is the solution of the system (1),(2); Θ is the set of unknown parameters. Denote

$$\min_{\theta\in\Theta} \Phi(\theta) = \epsilon.$$

Now it is possible to say that the system (1) is observable if $\epsilon = 0$, and if $\epsilon > 0$ then the system (1) is called $\epsilon-$ observable. In the case $\zeta(t) \equiv const$, it is possible to consider the following functional

$$\max_{t\in[t_0,t_1]} \max_{a\in[a_0,a_1]}[v(t) - y(a,t,\theta)] - \min_{t\in[t_0,t_1]} \min_{a\in[a_0,a_1]}[v(t) - y(a,t,\theta)]$$

which is eqvivalent to

$$\max_{t_1 t_2\in[t_0,t_1]} \max_{a_1 a_2\in[a_0,a_1]} [|v(t) - y(a_1,t_1,\theta)| - |v(t) - y(a_2,t_2,\theta)|]$$

Finally, is $|\zeta - \zeta_0| < \alpha < 0, \quad \zeta \in \Xi$, one can consider the following two functionals

1.
$$a) \quad \max_{t,a} |v(t) - y(a,t,\theta)| = \Phi(\theta) \le \alpha; \tag{4}$$
$$b) \quad \max_{t,a} |v(t) - y(a,t,\theta)| - \min_{t,a} |v(t) - y(a,t,\theta)| = \Phi(\theta) \le \alpha; \tag{5}$$

It is necessary to find a set Θ_α of elements satisfying (4) and (5).

Let us reduce the continuous system (1) to a discrete form.

Let $D = \{(a,t): \; 0 < a < A, \; 0 < t < T\}$. In the set D let us choose a grid with a step - size $\Delta t = \Delta a$.

Denote $M = [\frac{A}{\Delta t}] + 1$,

$u_k(a) = u(a, k\Delta t)$,

$u_k = \{ u_k(0), u_k(\Delta t), \ldots, u_k(M\Delta t) \}$.

Assuming that the functions μ and β are twice continuonsly differentiable in D, it is possible to linearize these functions

$$\mu(a,t,P(t),\theta) = \mu_0 + \mu_1 a + \mu_2 t + \mu_3 P(t) + \mu_4 \theta \ldots,$$

$$\beta(a,t,P(t),\theta) = \beta_0 + \beta_1 a + \beta_2 t + \beta_3 P(t) \ldots,$$

The size of the population at a moment t is

$$P_k = \Delta t \sum_{l=1}^{M} u_k(l\Delta t).$$

The sum of the partial derivatives with respeet to u is approximated in a characteristic direction by the difference

$$\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} \right) \Big|_{(k\Delta t, l\Delta t)} = \frac{(u_{k+1}((l+1)\Delta t) - u_k(l\Delta t))}{\Delta t}.$$

This implies

$$u_{k+1}((l+1)\Delta t) = \alpha_{kl} - \mu_4\theta + g_{kl}u_k \qquad l = \overline{0, M}$$

where

$$\alpha_{kl} = \mu_0 + f_0 + ((\mu_1 + f_1)l + (\mu_2 + f_2)k)\Delta t$$

$$g_{kl}^T = (\frac{-\mu_3}{2}(\Delta t)^2, -\mu_3\Delta t^2, \ldots, \underbrace{1 - \mu_3\Delta t^2}, \ldots, -\mu_3\Delta t^2, \ldots, \frac{-\mu_3}{2}(\Delta t)^2)$$

Thus, we have M out of $M+1$ componens of the vector u_{k+1}, expressed via the vector u_k. Now let us find a representation for the first component of the vector u_{k+1}.

$$u_{k+1}(0) = \sum_{l=1}^{M} \alpha_{k,l} B_{k+1,l} + \sum_{l=1}^{M} B_{k+1,l}(-\mu_4)\theta_k + \sum_{l=1}^{M}(-B_{k+1,l}g_{kl})u_k,$$

i.e.

$$u_{k+1} = A_k + G_k u_k + H_k\theta_k,$$

where

$$G_k = \begin{pmatrix} \sum_{l=1}^{M} B_{kl}g_{k1} & \sum_{l=1}^{M} B_{kl}g_{k2} & \cdots & \sum_{l=1}^{M} B_{kl}g_{k(M-1)} & \sum_{l=1}^{M} B_{kl}g_{k(Ml)} \\ -\frac{\mu_3}{2}\Delta t^2 & 1 - \mu_3\Delta t^2 & \cdots & -\mu_3\Delta t^2 & -\frac{\mu_3}{2}\Delta t^2 \\ -\frac{\mu_3}{2}\Delta t^2 & -\mu_3\Delta t^2 & \ddots & -\mu_3\Delta t^2 & -\frac{\mu_3}{2}\Delta t^2 \\ \vdots & \vdots & \ddots & & \vdots \\ -\frac{\mu_3}{2}\Delta t^2 & -\mu_3\Delta t^2 & \cdots & -\mu_3\Delta t^2 & 1 - \frac{\mu_3}{2}\Delta t^2 \end{pmatrix},$$

$$H_k = \left(\sum_{l=1}^{M} B_{k+1,l}(-\mu_4), -\mu_4, \ldots, -\mu_4 \right).$$

Recalling that the papulation is observable (see (2)), we obtain the following system related to the observation

$$y_{k+1} = G_k y_k + H_k \theta_k + \zeta_k$$

Let us discuss a more general problem.

Consider the following linear difference equation of the n-th order

$$x_{t+1} = \sum_{k=0}^{n-1} \alpha_k x_{t-k} + \sum_{k=0}^{m} \beta_k u_{t-k} + f_{t+1} \tag{6}$$

where α_k, β_k - are constant coefficients, f_t -is a sequence of independent Gaussian values with zero mean and the dispersion $\sigma^2; u_t$ - is a control, $\beta_0 \neq 0$. Assume that coefficients α_k are unknown. Denote

$$w_t = \sum_{k=0}^{m} \beta_k u_{t-k}, \qquad \theta^T = (\alpha_0, \alpha_1, ..., \alpha_{n-1}).$$

Then

$$x_{t+1} = \theta^T z_t + w_t + f_{t+1}, \tag{7}$$

where $z_t^T = (x_t, x_{t-1}, ..., x_{t-n})$. The transition function $P_\theta(x_{t+1}|z_t, w_t)$ of the process defined by the equation (6), take the form

$$P_\theta(x_{t+1}|z_t, w_t) = F(x_{t+1} - \theta^T z_t - w_t), \tag{8}$$

where $F(\cdot)$ -is the density of distribution of the random values f_t.

Consider the Bayes approach to evaluate the parameters of conditional distribution

$$P(x_{t+1}|\nu_t, z_t, w_t) = \int\limits_{\Theta} F(x_{t+1} - \theta^T z_t - w_t)\nu_t(\theta)n(d\theta),$$

$$\nu_{t+1}(\theta) = \nu_t(\theta)\frac{F(x_{t+1} - \theta^T z_t - w_t)}{P(x_{t+1}|\nu_t, z_t, w_t)}$$

Denote by $z_t^T = (x_t, x_{t-1}, ..., x_{t-n})$ the phase vector of the equation (6). Then this equation can be rewritten in the form

$$z_{t+1} = Az_t + b(\theta^T z_t + \sum_{k=0}^{m} \beta_k u_{t-k} + f_{t+1}), \tag{9}$$

where

$$A = \begin{pmatrix} 00 & \dots & 00 \\ 10 & \dots & 00 \\ \vdots & & \\ 00 & \dots & 01 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Theorem 1. For any T the strategy

$$\sum_{k=0}^{m} \beta_k u_{t-k} = -\theta_t^T z_t \tag{10}$$

minimizes the functional

$$J_T = E_\nu\{\sum_{t=0}^{T-1} z_t^T (I - S_T n) z_t + V(z_T)\}$$

and the relation

$$\inf_u J_T = nTd^2 + V(z_0)$$

holds.

Corollary 1. Let for some t the inequality $I - qS_T > \epsilon_0 I$. hold. Then there exists ρ, $0 < \rho < 1$, such that

$$E\{V(z_{t+1})|z_t, \nu_t\} \le \rho V(z_t) + nd^2.$$

Corollary 2. Let $\lambda(\theta)$ be an arbitrary density satisfying the inequality $\lambda(\theta) \le C\gamma(\theta)$. Then the following inequality

$$E_\lambda(\sum_t^T z_t^2) \le C E_\nu(\sum_t^T z_t^2)$$

hold.

Theorem 2. Let $\sum_{k=0}^{m} \beta_k u_{t-k} = -\theta_t^T z_t$ and $\lambda(\theta)$ be the density of a distribution concentrated in a bounded domain and satisfying the inequality $\lambda(\theta) < C\gamma(\theta)$. Then the following inequalities

$$E\{z_t^2\} \le const; \qquad E\{u_t^2\} \le const$$

hold.

Let us explain the theorem. The process $(x_t; \nu_t)$, defined by the transitive function $P(x_{t+1}|z_t, \nu_t, u_t)$ can be treated as a process defined by the equation

$$x_{t+1} = \theta_t^T z_t + \sum_{k=0}^{m} \beta^k u_{t-k} + f_{t+1},$$

where θ_t is a random vector with the distribution function $\lambda_t(\theta)$. Since the mean of the distribution $\lambda_t(\theta)$ is condensed in a neighbourhood of the point θ_0, then for *anyt* the inequality $|\theta_t - \theta_0| < \epsilon$ holds. Hence, Theorem 2 claims that the strategy (10) stabilizes a trajectory of equation (9) with a "smal" random perturbation of the vector of coefficients of equation (9). Naturally one can hope that this strategy stabilizes the trajectory and for the equation with the unperturbed vector of parameters $\theta_t \equiv \theta_0$.

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