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A MULTIDIMENSIONAL COMPLEX-ANALYTICAL VIEW ON THE MULTIPARAMETER SPECTRUM AND THE CONSTRUCTION OF THE SPECTRAL MEASURES

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Introduction

This paper concerns the multiparameter spectral (MPS) theory, which is related to the attempt to solve boundary value problems by the method of separation of variables. We identify this MPS problem with a suitable "spectral investigation" on the operator system

$$A(\lambda) = (A_1(\lambda), \ldots, A_n(\lambda)),$$

where

$$A_j(\lambda) = A_j - \lambda_1 B_{j1} - \ldots - \lambda_n B_{jn}$$

Let $A_j(\lambda)$ be an operator acting in a Hilbert space H_j and depending on "multidimensional" parameter $\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n$. It will be assumed that A_j is an unbounded (in general) self-adjoint operator and B_{jk} is a bounded self-adjoint operator for $j, k \in \{1, 2, \ldots, n\}$.
Let H be the Hilbert tensor product of the spaces $H_1, \ldots, H_n$. To each operator A_j and B_{jk} we associate the operator

$$A_j^t = I_1 \otimes \cdots \otimes I_{j-1} \otimes A_j \otimes I_{j+1} \otimes \cdots \otimes I_n$$

and B_{jk}^t acting in $H = H_1 \otimes \cdots \otimes H_n$, see [19]. The general method for studying the system $A(\lambda)$ consists in constructing the corresponding operators $\Delta_0, \Delta_1, \ldots, \Delta_n$ which are the (well-defined) determinants of the operator matrix $\left(B_{jk}^t\right)_{i,j=1}^n$ and the matrices obtained from this matrix by replacing the j-th column by that of operators $A_1^t, \ldots, A_n^t$. By definition we have

$$\Delta_0 = \sum_\sigma \varepsilon_\sigma B_{1\sigma(1)} \otimes \cdots \otimes B_{n\sigma(n)},$$

where $\sigma = (\sigma(1), \ldots, \sigma(n))$ runs through all permutations of (1,2,...,n) and ε_σ is the signature of σ. We can also introduce the other tensor determinants $\Delta_1, \ldots, \Delta_n$, defined by analogy with Δ_0. We note that the operator Δ_0 is bounded in H and the operators $\Delta_1, \ldots, \Delta_n$ admit closures.

We assume that Δ_0 is positive definite: $\Delta_0 >> 0$, i.e. $(\Delta_0 x, x) \geq \alpha(x, x)$ for some $\alpha > 0$ and for the arbitrary $x \in H$.

The separating system of operators $\Delta_0^{-1}\Delta_1, \ldots, \Delta_0^{-1}\Delta_n$ is the family associated with the multiparameter system $A(\lambda)$, and certain important problems in the MPS theory have a complete solution just because they can be expressed in terms of this family of operators.

The precise definitions along with various properties and the interconnection between the original MPS problems and the corresponding problems for the separating system of operators can be found in [4], [22], [15], [16], [13].

The **spectrum** of a multiparameter system $A(\lambda)$ is defined to be the set $\sigma[A(\cdot)]$ of all $\lambda \in C_n$ such that each of the operators $A(\lambda)$ is not invertible, see [11]. The point spectrum of $A(\lambda)$ is the set of $\lambda \in C_n$ such that each operator $A_j(\lambda)$ has a nonzero kernel. Let us note the following important properties of self-adjoint MPS systems which are well known from the standard multiparameter theory, see [8], [21], [22], [12], [14]. The separating system of operators $\Gamma_j = \Delta_0^{-1}\Delta_j$, $j = 1, 2, \ldots, n$ consists of essentially self-adjoint operators (i.e., the closure $\overline{\Gamma_j}$ is self-adjoint) in the space $\langle H \rangle$, which is the Hilbert tensor product $H_1 \otimes \cdots \otimes H_n$ with the "weight" inner product $\langle x, y \rangle = (\Delta_0 x, y)$. The operators $\overline{\Gamma_1}, \ldots, \overline{\Gamma_n}$ are pairwise commuting in the sense that their spectral measures $E_{\overline{\Gamma_1}}, \ldots, E_{\overline{\Gamma_n}}$ commute. Let E_Δ denote the standard spectral measure $E_{\overline{\Gamma_1}} \otimes \ldots \otimes E_{\overline{\Gamma_n}}$ of the strongly commuting family of self-adjoint operators $\overline{\Gamma_1}, \ldots, \overline{\Gamma_n}$. Further, we have

$$\sigma[A(\cdot)] = \operatorname{Supp} E_\Delta \stackrel{\text{def}}{\equiv} \sigma^{jt}(\overline{\Gamma_1}, \ldots, \overline{\Gamma_n}).$$

Here the left hand side is the spectrum of the MPS system $A(\lambda)$ and the right hand side is the joint spectrum of the strongly commuting separating system of self-adjoint operators.

Then it is natural to call joint spectral measure E_Δ of the separating system $\overline{\Gamma_1}, \ldots, \overline{\Gamma_n}$ the **spectral measure of the self-adjoint MPS problem for the system of operators** $A(\lambda)$.

This paper deals with the geometrical and analytical structure of the spectrum $\sigma[A(\cdot)]$ and the construction of the spectral measures of the self-adjoint MPS problem beginning with the corresponding measures of the original self-adjoint operators $A_j(\lambda)$, $\lambda \in R^n$. Further, in addition the operators $A_1, \ldots, A_n$ are assumed to have compact resolvents except one. From the point of view of applications in mathematical physics these requirements can be regarded as natural (one radial and several angular variables arise by applying the method of separation of variables.)

In the fifties H.O. Cordes published a series of papers on the method of separation of variables studied in the Hilbert space. See [9], [10] also [18]. The solution of the problem for some two-parameter operator systems can be deduced from these works by Cordes (n=2, $A = A_1^*$ is arbitrary and $A_2 = A_2^*$ has a discrete spectrum, $B_{11} + B_{12} = I$, $-B_{21} + B_{22} = I$, $B_{11} \geq 0$, $B_{12} \geq 0$, $B_{21} \leq 0$, $B_{22} \geq 0$, $\Delta_0 >> 0$- consequently, the operators B_{j1} and B_{j2} commute).

For the three-parameter case see [1] and for some general discussion see [2]. We essentially use the Bishop's ideas (see [7]) concerning the structure of the roots of analytic functions of several complex variables with values in Banach space and some arguments of the geometric theory of functions of several complex variables. To construct a spectral measure in a general n-parameter problem we essentially use the Cordes method for the two-parameter case.

§1. Input Multiparameter Operators Representation in terms of a Separating System

Let B_{jk} be self-adjoint bounded operators and A_j be a self-adjoint unbounded operator in a Hilbert space H_j, $j \in \{1, 2, \dots, n\}$, and $H = H_1 \otimes \dots \otimes H_n$.

Further, denote

$$\Delta_0 = \det\left(B_{jk}^t\right) = \begin{vmatrix} B_{11} & \dots & B_{1n} \\ \dots & \dots & \dots \\ B_{n1} & \dots & B_{nn} \end{vmatrix}, \qquad (1)$$

and let Δ_j be a tensor determinant operator in H which can be defined in the usual way, namely by replacing the j-th column of Δ_0 by the column of operators $A_1, \dots, A_n$. For example, if n=2, we have

$$\Delta_0 = B_{11} \otimes B_{22} - B_{12} \otimes B_{21} \quad \text{and} \quad \Delta_1 = A_1 \otimes B_{22} - B_{12} \otimes A_2.$$

By definition we set

$$D(\Delta_j) = D(A_1) \underset{a}{\otimes} D(A_2) \otimes \dots \otimes D(A_n),$$

where $\otimes$ is the algebraic tensor product. If $\Delta_0 >> 0$, then the operators $\Gamma_j = \Delta_0^{-1} \Delta_j$, $j = 1, 2, \dots, n$ are essentialy self-adjoint operators in a new Hilbert space $\langle H \rangle$ with an inner product

$$\langle \cdot, \cdot \rangle = (\cdot, \Delta_0 \cdot).$$

Let us introduce the operators

$$B_{jk}(\nu)=\sum_{m=1}^{n} B_{jm} f_{mk}(\nu), \quad j,k=1,2,\dots,n \tag{2}$$

depending upon the variable ν, that is,

$$\left(B_{jk}(\nu)\right)_{n\times n}=\left(B_{jk}\right)_{n\times n}\left(f_{jk}(\nu)\right)_{n\times n},$$

where $f_{jk}(\nu)$ are some scalar functions to be determined later. Now we determine

$$\Gamma_j(\lambda,\nu)=\sum_{k=1}^{n}\left(\Gamma_k-\lambda_k\right) g_{jk}(\nu), \quad j=1,2,\dots,n,$$

where $g_{jk}(\nu)$ is a cofactor of the element $f_{jk}(\nu)$ of the matrix $\left(f_{jk}\right)_{n\times n}$. Then

$$\begin{vmatrix} A_1(\lambda) & B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & & \cdots \\ A_n(\lambda) & B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} = \Delta_1 g_{11}+\Delta_2 g_{12}+\dots+\Delta_n g_{1n}-$$

$$-\Delta_0\left(\lambda_1 g_{11}+\lambda_2 g_{12}+\dots+\lambda_n g_{1n}\right)=\left(\Delta_1-\lambda_1\Delta_0\right)g_{11}+\dots+\left(\Delta_n-\lambda_n\Delta_0\right)g_{1n},$$

and hence

$$\Gamma_1(\lambda,\nu)\,x=\Delta_0^{-1}\begin{vmatrix} A_1(\lambda) & B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ A_n(\lambda) & B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} x. \tag{31}$$

In a similar manner we have

$$\Gamma_n(\lambda,\nu)x=\Delta_0^{-1}\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu) & A_1(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{n,n-1}(\nu) & A_n(\lambda) \end{vmatrix} x, \tag{3n}$$

for $x \in D(A_1)\otimes\ldots\otimes D(A_n)$. Now multiplying (3j) by $B^t_{1j}(\nu)=B_{1j}(\nu)\otimes I_2\otimes\ldots\otimes I_n$ and summing up we obtain

$$\begin{aligned}
\sum_{j=1}^{m} B_{1j}(\nu)\Gamma_j(\lambda,\nu)x = \{ & B^t_{1j}(\nu)\Delta_0^{-1}\begin{vmatrix} B_{22}(\nu) & \cdots & B_{2n}(\nu)\\ \cdots & \cdots & \cdots \\ B_{n2}(\nu) & \cdots & B_{nn}(\nu)\end{vmatrix} - \\
& B^t_{12}(\cdot)\Delta_0^{-1}\begin{vmatrix} B_{21}(\nu) & B_{23}(\nu) & \cdots & B_{2n}(\nu)\\ \cdots & \cdots & \cdots & \cdots \\ B_{n1}(\nu) & B_{n3}(\nu) & \cdots & B_{nn}(\nu)\end{vmatrix} + \ldots + \\
& +(-1)^{n-1}B^t_{1n}(\nu)\Delta_0^{-1}\begin{vmatrix} B_{21}(\nu) & \cdots & B_{2,n-1}(\nu)\\ \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{n,n-1}(\nu)\end{vmatrix}\}A^t_1(\lambda)x + \\
& +\left\{-B^t_{11}(\nu)\Delta_0^{-1}\begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu)\\ B_{32}(\nu) & \cdots & B_{3n}(\nu) \\ \cdots & \cdots & \cdots\end{vmatrix} + B^t_{12}(\nu)\Delta_0^{-1}\cdot\right. \\
& \cdot\begin{vmatrix} B_{11}(\nu) & B_{13}(\nu) & \cdots \\ B_{32}(\nu) & B_{3n}(\nu) & \cdots \\ \cdots & & \cdots\end{vmatrix} + \ldots + (-1)^n B^t_{1n}(\nu)\Delta_0^{-1}\cdot \\
& \begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu)\\ B_{31}(\nu) & \cdots & B_{3,n-1}(\nu) \\ \cdots & \cdots & \cdots\end{vmatrix}\}A^t_2(\nu)x + \ldots + \\
& +\{B^t_{11}(\nu)\Delta_0^{-1}\begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu)\\ \cdots & \cdots & \cdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu)\end{vmatrix} - \ldots + \\
& +(-1)^{n-1}B^t_{1n}(\nu)\Delta_0^{-1}\cdot\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu)\\ \cdots & \cdots & \cdots \\ B_{n-1,1}(\nu) & \cdots & B_{n-1,n-1}(\nu)\end{vmatrix}\}A^t_n(\lambda)x. \qquad (4)
\end{aligned}$$

Let us assume that

$$\det\left(f_{jk}(\nu)\right)_{n\times n}=1.$$

It is easy to show that

$$\Delta_0=\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1n}(\nu)\\ \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{nn}(\nu)\end{vmatrix}.$$

Then the expression in the first bracket on the right hand side of (4) equals to 1 and the others equal to 0. Thus, by analogy for the other sums $\sum_j B^{t}_{kj}(\nu)\Gamma_j(\lambda,\nu)$ we obtain the relations

$$A^{t}_{k}(\lambda)x=\sum_{m=1}^{n}B^{t}_{km}(\nu)\Gamma_m(\lambda,\nu)x\,,\quad k=1,2\ldots n \qquad (5)_1$$

for

$$x\in D(A_1)\underset{a}{\otimes}\ldots\otimes D(A_n).$$

In particular, if $\lambda=(0,\ldots,0)$ and $f_{11}(\nu)=\ldots=f_{nn}(\nu)=1$, $f_{jk}(\nu)=0$, for $j\neq k$, we obtain

$$A^{t}_{j}x=\sum_{k=1}^{n}B^{t}_{jk}\overline{\Gamma_k}x\,,\;\; x\in D(A_1)\underset{a}{\otimes}\ldots\otimes D(A_n). \qquad (5)_2$$

According to the MPS theory of self-adjoint operators (see [8], [23], [12], [14], [22]), we have

$$\bigcap_{j=1}^{n}D\left(\Gamma_j(\lambda,\nu)\right)=\bigcap_{j=1}^{n}D\left(A^{t}_{j}(\lambda)\right),\quad \lambda\in R^{n},$$

(let us recall that the operator $A^{t}_{j}(\nu)$ is closed by definition).

Let $x \in \bigcap_j D(\Gamma_j)$. Then there exists the sequence $(x_n)_1^\infty \subset D(A_1) \otimes_a \ldots \otimes D(A_n)$ such that $x_n \to x$ and

$$A_j^t x_n \to A_j^t x, \quad j = 1, 2, \ldots, n.$$

In fact, $D\left(A_j^t\right) = D\left(\left|A_j^t\right|\right)$ that is why, the operator $\left|A_1^t\right| + \ldots + \left|A_n^t\right|$ is determined in $D\left(A_1^t\right) \cap \ldots \cap D\left(A_n^t\right)$. If $x_n \to x$ and $\left(\left|A_1^t\right| + \ldots + \left|A_n^t\right|\right) x_n \to \left(\left|A_1^t\right| + \ldots + \left|A_n^t\right|\right) x$, then we have

$$0 \leftarrow \left\| \left(\left|A_1^t\right| + \ldots + \left|A_n^t\right| \right)(x_n - x) \right\|^2 = \left\| \left|A_1^t\right| (x_n - x) \right\|^2 +$$

$$+ \left\| \left(\left|A_2^t\right| + \ldots + \left|A_n^t\right| \right)(x_n - x) \right\|^2 + 2\left(\left|A_1^t\right| (x_n - x), \left(\left|A_2^t\right| + \ldots + \left|A_n^t\right| \right)(x_n - x) \right).$$

The family of the operators $\left|A_j^t\right|$, $j = 1, 2, \ldots, n$ is commutative, so the last term of this sum is a non-negative number. Hence

$$\left|A_1^t\right| (x_n - x) \to 0.$$

We have

$$A_1^t (x_n - x) \to 0.$$

Thus, $A_j^t x_n \to A_j^t x, \quad j = 1, 2, \ldots, n$

Taking $A_j^t(\lambda)$ instead of A_j^t we obtain the same proof for $A_j^t(\lambda)$, $j = 1, \ldots n$ and $\lambda \in R^n$.

Further, from $A_j^t(\lambda) x_n \to A_j^t(\lambda) x$ it follows that

$$\Delta_j(\lambda,\nu)\,x_n \to \overline{\Delta}_j(\lambda,\nu)\,x, \quad j=1,2,\ldots,n,$$

where $\Delta_j(\lambda,\nu)$ is obtained from Δ_j by replacing A_j by $A_j(\lambda)$ and B_{jk} by $B_{jk}(\nu)$. Taking into account that Δ_0^{-1} is a bounded operator we obtain

$$\Gamma_j(\lambda,\nu)\,x_n \to \Gamma_j(\lambda,\nu)\,x, \quad j=1,2,\ldots,n$$

Thus,

$$A_j^t(\lambda)\,x = \sum_{k=1}^{n} B_{jk}^t(\nu)\,\overline{\Gamma_k(\lambda,\nu)}\,x$$

for each element

$$x \in \bigcap_j D\left(A_j^t(\lambda)\right) = \bigcap_j D\left(A_j^t\right).$$

This proves the following proposition:

Lemma.1 Let A_j and B_{jk} be self-adjoint operators and $\Delta_0 >> 0$ and $B_{jk}(\nu) = \sum_m B_{jm} f_{jk}(\nu)$, $j,k=1,2,\ldots,n$, where $f_{jk}(\nu)$ are some scalar functions such that

$$\det\left(f_{jk}(\nu)\right)_{n\times n} = 1,$$

then the following relation

$$A_j^t(\lambda)\,x = \sum_k B_{jk}^t(\nu)\,\Gamma_k(\lambda,\nu)\,x, \quad j=1,2,\ldots,n \qquad \ldots(5)$$

holds for each $x \in \bigcap_{j=1}^{n} D\left(A_j^t\right)$.

§2. n-1 Discrete Problems Structure with n-Parameters

Let $A_1, A_2, \dots, A_{n-1}$ be operators with a discrete spectrum (that is, their resolvents are compact operators) and the following conditions be satisfied:

$$\delta_{jk} \cdot B_{jk} \gg 0, \quad j = 1,2,\dots,n-1, \quad k = 1,2,\dots,n \quad \text{for some } \delta_{jk} = \pm 1, \tag{6}$$

$$\varepsilon_k \begin{vmatrix} B_{11} & \cdots & B_{1,k-1} & B_{1,k+1} & \cdots & B_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n-1,1} & \cdots & B_{n-1,k-1} & B_{n-1,k+1} & \cdots & B_{n-1,n} \end{vmatrix} \gg 0 \tag{7}$$

for some set of sign factors $\varepsilon_k = \pm 1$.

We note at once that the formulas (6) and (7) do not impose essential restrictions on the operators B_{jk} in the sense of following propositions:

Lemma 2. If $\Delta_0 \gg 0$, then the operators B_{jk} can be replaced by their non-degenerate linear combinations such that these new operators satisfy the conditions (6) and (7).

Proof: Let $x_0^n \in H_n$, such that $\left(B_{nn} x_0^n, x_0^n\right) \neq 0$. Then

$$\begin{vmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \vdots & \vdots \\ B_{n-1,1} & \cdots & B_{n-1,n} \\ \left(B_{n1} x_0^n, x_0^n\right) & \cdots & \left(B_{nn} x_0^n, x_0^n\right) \end{vmatrix} = \left(B_{nn} x_0^n, x_0^n\right) \cdot \det_{\otimes} \left(B_{jk} - \frac{\left(B_{nk} x_0^n, x_0^n\right) \cdot B_{jn}}{\left(B_{nn} x_0^n, x_0^n\right)} \right)_{(n-1)\times(n-1)}$$

Assume that $\left(B_{nn} x_0^n, x_0^n\right) > 0$ for the sake of simplicity. Then B_{jk} can be replaced by

B'_{jk}, where

$$B'_{jk} = B_{jk} - \frac{\left(B_{nk} \overset{n}{x_0}, \overset{n}{x_0}\right)}{\left(B_{nn} \overset{n}{x_0}, \overset{n}{x_0}\right)} B_{jn}, \qquad B'_{jn} = B_{jn}, \qquad j = 1,2,\dots,n-1, \quad k = 1,2,\dots,n$$

and we have

$$\otimes \det \left(B'_{jk}\right)_{j,k=1}^{n} \gg 0.$$

For every set of sign factors $\varepsilon_r = \pm 1$, $r = 1,2,\dots,n-1$ there is a non-zero vector $\alpha(\varepsilon) \in R^{n-1}$ such that

$$\varepsilon_r \sum_{s=1}^{n-1} \alpha_s(\varepsilon) B_{rs} \gg 0, \qquad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n),$$

see [5].
We set

$$B''_{j1} = \sum_{s=1}^{n-1} \alpha_s(I) B'_{js}, \quad I = (1,1,\dots,1)$$

$$B''_{jk} = B'_{jk} + \ell B'_{j1} \quad, j = 1,2,\dots,n; \quad k = 1,2,\dots,n-1,$$

where ℓ is a large enough number. Taking $B'_{jn} + c\sum_{k=1}^{n} B''_{jk}$, instead of B'_{jk}, where c is a large enough number we get the formulas (6) and (7).
This proves lemma 2.
Now, let us consider another two-parameter operator

$$A(\lambda_1, \lambda_2) = A - \lambda_1 B_1 - \lambda_2 B_2,$$

where A is an arbitrary self-adjoint operator with a discrete spectrum and B_1, B_2 are self-adjoint bounded operators, moreover,

$$B_1 \gg 0, \quad B_2 \gg 0 .$$

Lemma 3. The real spectrum of the operator $A(\lambda_1,\lambda_2)$ consists of eigenvalues only and we have

$$\sigma[A(\cdot)]\cap R^2=\sigma_p[A(\cdot)]\cap R^2=\bigcup_{n=1}^{\infty}\gamma_n,$$

where γ_n is the analytic curve in R^2. Moreover, the points of intersection do not accumulate in the finite part of R^2 and if $\gamma_n=\{\lambda:\lambda_2=\varphi_n(\lambda_1)\}$, then we have

$$\frac{d\varphi_n}{d\lambda_1}\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)=-\frac{(B_1u,u)}{(B_2u,u)},$$

for an arbitrary $u=\mathrm{Ker}\left(A\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)\right)$, provided $\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)\in\gamma_n\cap\gamma_{n'}$, $(n\neq n')$.

Proof. It is clear that if $\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)\in\sigma\cap R^2$, then $\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)\in\sigma\left[B_2^{-1}A(\cdot)\right]$. Since the operator $B_2^{-1}A$ has a discrete spectrum, the same is true for the operator $B_2^{-1}A-\overset{0}{\lambda}_2$. According to the well-known theorem of the perturbation theory (see [17], theorem VII.1.8 and II.1.10) the spectrum in some neighbourhood of the point $\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)$ consists of the analytic curves γ'_k, $k=1,2,\ldots,m'$ passing through $\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)$ and for every curve γ'_k we have

$$\left.\frac{d\varphi_k}{d\lambda_1}\right|_{\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)}=-\left(B_2^{-1}B_1u,u\right)_{B_2}=\frac{(B_1u,u)}{(B_2u,u)},\tag{8}$$

where $u=\mathrm{Ker}\left[B_2^{-1}A\left(\overset{0}{\lambda}_1,\overset{0}{\lambda}_2\right)\right]$ and $(x,y)_{B_2}=(x,B_2y)$ (the formula (8) is proved by Rellich, see [20]).

Since $B_2^{-1}B_1$ is a strongly positive operator on H_{B_2}, we have

$$a\leq\frac{d\varphi_k}{d\lambda_1}\leq b,$$

where a and b are some negative numbers.
Then each of the function φ is continued along the whole R analytically.
Indeed, if

$$\gamma' = \{\lambda : \lambda_2 = \varphi(\lambda_1)\},$$

and $\lambda_1' = \partial(\mathrm{pr}_{o\lambda_1}\gamma')$ is a boundary point of projection of the curve γ' on the axis $\Phi\lambda_1$, then the function φ is continued through λ_1' into some of its neighbourhood, because all spectrum points in some neighbourhood of $\left(\overset{0}{\lambda}_1, \overset{0}{\lambda}_2\right) \in \partial\gamma'$ consist of a finite number of analytic curves and it is clear that one of them is a continuation of φ. Similarly, if

$$\mu_1 = \operatorname{Sup} \lambda_1, \ \ \mu_1' = \inf \lambda_1,$$

where Sup and inf are taken with respect to the set of those λ, in which φ is continued, then φ is continued through μ_1 into some of its neighbourhood. Thus, we have $\mu_1 = \infty, \quad \mu_1 = -\infty$ and lemma 3 is proved.

Lemma 4. The set $\sigma[A_1(\lambda)] \cap R^n$ consists of at most countable number of the analytic surfaces

$$\wp_m = \{\lambda : \lambda_n = \varphi_m(\lambda_1, \ldots, \lambda_{n-1})\}$$

(φ_m is the analytic function in R^{n-1}).Only a finite number of surfaces can pass through each point $\lambda \in R^n$

Proof. It is known that $\sigma[A_1(\lambda)]$ is the complex analytic set (see [7]). Assume that $\overset{0}{\lambda} \in \sigma[A_1(\lambda)] \cap R^n$. There exist non-zero functions $F_m(\lambda_1, \lambda_2 \ldots, \lambda_n)$, holomorphic in some complex neighbourhood U of the point $\overset{0}{\lambda}$ such that common zeros of these functions coincide with

$$\sigma[A_1(\lambda)] \cap U, \quad m = 1, 2, \ldots, r.$$

First of all, let us consider a zero-set (the set of all roots) of the function F_1. We denote $\hat{\lambda}=(\lambda_1,\dots,\lambda_{n-1})$. Suppose that $F_1\left(\hat{\lambda}^0,\lambda_n\right)\neq 0$ (this condition can always be obtained by the linear combination of the variables). Then by the Weierstrass theorem (see [21], §8, VI) in some neighbourhood of $\overset{0}{\lambda}$ (without loss of the generality in U) the function F_1 can be represented in the form:

$$F_1(\lambda)=\left\{\left(\lambda_n-\overset{0}{\lambda}_n\right)^k+C_1\left(\hat{\lambda}\right)\left(\lambda_n-\overset{0}{\lambda}_n\right)^{k-1}+\dots+C_k\left(\hat{\lambda}\right)\right\}\varphi_0(\lambda),$$

where C_m are holomorphic in $\hat{U}=\left\{\left(\lambda_1,\dots,\lambda_{n-1}\right):\left(\lambda_1,\dots,\lambda_n\right)\in U\right\}$, $C_m\left(\hat{\lambda}^{\circ}\right)=0$ and $\varphi_0(\lambda)\neq 0$ for $\lambda\in U$.

Thus, zeros of F_1 are given by the equation

$$P(\lambda)=\left(\lambda_n-\overset{0}{\lambda}_n\right)^k+C_1\left(\hat{\lambda}\right)\left(\lambda_n-\overset{0}{\lambda}_n\right)^{k-1}+\dots+C_k\left(\hat{\lambda}\right)=0.$$

This equation has k number of roots with respect to λ_n:

$$\lambda_n^{(N)}=g_N^1\left(\hat{\lambda}\right),\quad N=1,2,\dots k,$$

where the function g_N^1 are locally holomorphic in $\hat{U}$ everywhere except the set $\mathcal{A}_1$, in which the equation has at least one multiple root. Indeed, we have $\frac{\partial P}{\partial\lambda_n}\neq 0$ for the $\hat{\lambda}\in\hat{U}\setminus\mathcal{A}_1$ and it is sufficient to apply the implicit function theorem. In a similar manner zeros of each function F_m are given by the locally holomorphic functions g_N^m of the type g_n^1.

Assume that $(\mu_1,\mu_2,\dots,\mu_n)\in\sigma[A_1(\lambda)]\cap R^n$ and

$$(\mu_1,\mu_2,\dots,\mu_{n-1})\in\hat{U}\setminus\bigcup_{m=1}^{r}\mathcal{A}_m,$$

where $\mathcal{A}_m$ is the analytic set, where the function g_N^m may

be of non-holomorphic character.
Then there exists some neighbourhood $\hat{U}(\mu_1,\mu_2,\ldots,\mu_{n-1})$ such that all the functions g_n^k are holomorphic.
Denote

$$\wp'_{N,k}=\left\{\lambda:\hat{\lambda}\in\hat{U}(\mu_1,\mu_2,\ldots,\mu_{n-1}),\ \ \lambda_n=g_N^k(\lambda_1,\ldots,\lambda_{n-1})\right\}.$$

Then the part of the set $\sigma[A_1(\lambda)]\cap R^n$ which is in the neighbourhood U_μ can be represented as a union of all the various intersections

$$\left(\wp'_{N,k}\cap\ldots\cap\wp'_{N,k}\right)\cap U_\mu\cap R^n$$

We shall prove that

$$\sigma[A_1(\lambda)]\cap R^n\cap\ \ U_\mu=\bigcup_{n_k}\left(\wp'_{n_k,1}\cap R^n\right), \tag{9}$$

that is, the unit of some surfaces (which corresponds to zeros of the only function F_1) coincides with the spectrum in the neighbourhood $U_\mu\cap R^n$.

To prove it, let us consider the simple case.
Let the number of analytic functions F_j be equal to 2 and let each of the function F_j; $j=1,2$ have two corresponding different surfaces in R^n, namely $\wp_1$ and $\wp_2$ for F_1, also Q_1 and Q_2 for F_2. Then (9) means that the set $\sigma[A_1(\lambda)]\cap R^n\cap U_\mu$ coincides with one of $\wp_1$, $\wp_2$ or $\wp_1\cup\wp_2$. Indeed, let $\wp_j$ and Q_j be determined correspondingly in terms of the functions

$$\lambda_n=P_j(\lambda_1,\lambda_2\ldots,\lambda_{n-1})\ \text{and}\ \lambda_n=q_j(\lambda_1,\ldots,\lambda_{n-1}).$$

Let us investigate three cases:
1^0. If $\wp_j\neq Q_k$, for all j, k=1,2, then

$$\sigma[A_1(\lambda)]\cap R^n\cap U_\mu=\bigcup_{j,k}\left(\wp_j\cap Q_k\right).$$

It is clear that $\wp_j \cap Q_k$ is a curve in R^n.

We recall that the curve $\gamma \subset R^n$ satisfies the following condition:

in the neighbourhood of each point $(\lambda_1,\dots,\lambda_{n-1}) \in R^{n-1}$ there is a point $(\xi_1,\dots,\xi_{n-1})$ such that $(\xi_1,\dots,\xi_{n-1},\xi_n) \notin \gamma$ for all $\xi_n \in R$.

Let prove that the point $\xi \in \sigma[A_1(\lambda)] \cap R^n \cap U_\mu$ does not satisfy the last condition. First we consider the following two-parameter operator with respect to λ_{n-1},λ_n.

$$A_1(\lambda_{n-1},\lambda_n) = \left(A_1 - \xi_1^0 B_{11} - \dots - \xi_{n-2}^0 B_{1n-2}\right) - \lambda_n B_{1n-1} - \lambda_n B_{1n}.$$

Then $\left(\xi^0_{n-1},\dots,\xi^0_n\right)$ belongs to the last operator spectrum. According to Lemma 3 some analytic curve $\lambda_n = \varphi(\lambda_{n-1})$ passing through $\left(\xi^0_{n-1},\dots,\xi^0_n\right)$ also belongs to this one and the equation

$$\left.\frac{d\lambda_n}{d\lambda_{n-1}}\right|_{(\lambda_{n-1},\lambda_n)} = \frac{\left(B_{1,n-1}u^1, u^1\right)}{\left(B_{1,n}u^1, u^1\right)}$$

holds, where $u^1 \in \mathrm{Ker}\left(A_1 - \xi_1^0 B_{11} -, \dots, -\xi^0_{n-2} B_{1n-2} - \lambda_{n-1} B_{1n-1} - \lambda_n B_{1n}\right)$.

Assume that $\left(\lambda^1_1,\dots,\lambda^1_{n-1}\right) \in \hat{U}'$, where $\hat{U}'$ is a small enough neighbourhood of the point $\left(\xi^0_1,\dots,\xi^0_{n-1}\right)$.

Denote $\prod = \left\{\lambda; \lambda_{n-1} = \lambda^0_{n-1}\right\}$, $\gamma_2 = \left\{\lambda: \lambda_1 = \xi^0_1,\dots,\lambda_{n-2} = \xi^0_{n-2}, \lambda_n = \varphi(\lambda_{n-1})\right\}$ and $\prod \cap \gamma_2 = \lambda'' = \left(\xi^0_1,\dots,\xi^0_{n-2},\lambda^1_{n-1},\varphi\left(\lambda^1_{n-1}\right)\right)$.

Assume that our proposition holds for n−1 parameters. Let $\left(\mu^0_1,\dots,\mu^0_{n-1}\right) \in \sigma\left[A^1_1 - \lambda_1 B^1_{11} - \dots - \lambda_{n-1} B^1_{1,n-1}\right] \cap R^{n-1}$, where A^1_1 is self-adjoint and B^1_{jk} are bounded, strongly positive or negative operators. For each point

$\left(\eta_1,\ldots,\eta_{n-2}\right)$ from a small enough neighbourhood of $\left(\overset{0}{\mu}_1,\ldots,\overset{0}{\mu}_{n-1}\right)$ there exists $\eta_{\nu-1}\in R$ such that

$$\left(\eta_1,\ldots,\eta_{n-1}\right)\in\sigma\left[\overset{1}{A}_1-\lambda_1\overset{1}{B}_{11}-\ldots-\lambda_{n-1}\overset{1}{B}_{1,n-1}\right]\cap R^{n-1}.$$

We can apply this argument for the operator

$$\left(\overset{1}{A}_1-\overset{1}{\lambda}_{n-1}\overset{1}{B}_{1,n-1}\right)-\lambda_1\overset{1}{B}_{11}-\ldots-\lambda_{n-2}\overset{1}{B}_{1,n-2}-\lambda_n\overset{1}{B}_{1n}.$$

If $\left(\overset{0}{\xi}_1,\ldots,\overset{0}{\xi}_{n-2},\varphi\left(\overset{1}{\lambda}_{n-1}\right)\right)$ belongs to the last operator-functions spectrum, then it follows that for the point $\left(\overset{1}{\lambda}_1,\ldots,\overset{1}{\lambda}_{n-2}\right)$ there exists $\overset{1}{\lambda}_n\in R$ such that

$$\left(\overset{1}{\lambda}_1,\ldots,\overset{1}{\lambda}_{n-1},\overset{1}{\lambda}_n\right)\in\sigma\left[\overset{1}{A}_1-\lambda_1\overset{1}{B}_{11}-\ldots-\lambda_{n-2}\overset{1}{B}_{1,n-2}-\overset{1}{\lambda}_{n-1}\overset{1}{B}_{1,n-1}-\lambda_n\overset{1}{B}_{1,n}\right]$$

or

$$\left(\overset{1}{\lambda}_1,\ldots,\overset{1}{\lambda}_{n-2},\overset{1}{\lambda}_{n-1},\ \overset{1}{\lambda}_n\right)\in\sigma\left[\overset{1}{A}_1(\lambda)\right].$$

2^0 $\wp_1=Q_1$ and $\wp_2\neq\overset{(1)}{Q}_2$

If $\wp_2\cap Q_2\not\subset\wp_1$, then by repeating the previous arguments we shall have a contradiction.

3^0. If $\wp_1=Q_1$ and $\wp_2=Q_2$, then we obtain

$$\sigma\left[A_1(\lambda)\right]\cap U_\mu=\wp_1\cup\wp_2.$$

Thus, $\sigma\left[A_1(\lambda)\right]\cap U_\mu$ consists of some surfaces $\overset{1}{\wp}_1\cup\ldots\cup\overset{1}{\wp}_\ell$, where we denote $\overset{1}{\wp}_k=\overset{1}{\wp}_{N_k-1},\ k=1,\ldots,\ell$ for simplicity.

For each surface $\wp_k^1$ there exists some analytic function g_k such that we have $\lambda_n = g_k(\lambda_1,\ldots,\lambda_{n-1})$ for the points $\lambda \in \wp_k^1$. Let us prove that g_k has the analytic continuation on all R^{n-1}.

If $\lambda^{(1)} \in \partial\wp_k^1 \setminus A_1$ (let us recall that the analytic set is closed and does not divide any domain), then $\lambda^{(1)} \in \sigma[A_1(\lambda)]$ (as the spectrum is closed). By repeating the previous arguments for $\lambda^{(1)}$ we obtain that in some its neighbourhood all points of the spectrum belong to some analytic surfaces Q_m^1, $m = 1,2,\ldots^{\,l}$. Each surface Q_m^1 is given by the equation $\lambda_n = q_m(\lambda_1,\ldots,\lambda_{n-1})$, where q_m is an analytic function.

Let $P_r(\lambda_1,\ldots,\lambda_{n-1})P_k^1$ denote the projection of $\wp_k^1$ on the hyperplane $(\lambda_1,\ldots,\lambda_{n-1})$. It is clear that $\left(P_r(\lambda_1,\ldots,\lambda_{n-1})\wp_k^1\right) \cap \hat{U}_1$ is an open set. If $g_k(\lambda_1,\ldots,\lambda_{n-1}) \neq q_m(\lambda_1,\ldots,\lambda_{n-1})$ for all $m = 1,2,\ldots l^1$ and $(\lambda_1,\ldots,\lambda_{n-1}) \in \left(P_r(\lambda_1,\ldots,\lambda_{n-1})\wp_k^1\right) \cap U_1$, then the whole set $\wp_k^1 \cap U_1$ cannot be covered by the union $\bigcup_{k,m}\left(\wp_k^1 \cap Q_m^1\right)$ (see the above mentioned arguments). So we have $g_k(\lambda_1,\ldots,\lambda_{n-1}) = q_m(\lambda_1,\ldots,\lambda_{n-1})$ for some m and for all $(\lambda_1,\ldots,\lambda_{n-1}) \in \left[P_r(\lambda_1,\ldots,\lambda_{n-1})\wp_k^1\right] \cap \hat{U}_1$.

Then the function g_k is analytically continued through the points $\lambda^{(1)} \in \partial\wp_k$ (according to the definition of the analytic continuation).

Assume that $\left(\hat{\lambda}_1^1,\ldots,\hat{\lambda}_{n-1}^1\right) \in \hat{U}_1 \setminus A_1$ By definition of the analytic set there exists the curve $\hat{\lambda}^{(1)}\hat{\lambda}^1 \subset \hat{U} \setminus A_1$, and if $\hat{\lambda}^{(2)} = \left(\hat{\lambda}^{(1)}\hat{\lambda}^1\right) \cap \partial\left[P_r(\lambda_1,\ldots,\lambda_{n-1})\wp_k^1 \cap Q_m^1\right]$, then there exists $\lambda^{(2)} \in \partial\left(\wp_k^1 \cap Q_m^1\right)$ such that $\lambda^{(2)} = \left(\hat{\lambda}^{(2)}, \hat{\lambda}_n^2\right)$, see[21].

Thus, the function g_k is continued through $\hat{\lambda}^{(2)}$ in a similar manner.Let M be the set of those points of $\hat{\lambda}^{(1)}\hat{\lambda}^1$, on which the function g_k is continued in this way.

It is easy to see that M is at the same time closed and open, that is, $M=\hat{\lambda}^{(1)}\hat{\lambda}^{1}$. Indeed, the spectrum is closed and the function is continued from the neighbourhood into the neighbourhood.

Thus, g_k is continued into the whole $\hat{U}\setminus \mathcal{A}_1$ analytically.

According to the well-known theorem of the theory of several complex variables (see[21], theorem 3, §10) if the function is holomorphic in some domain, except some analytic set of the co-dimension one and locally bounded in $\mathcal{A}_1$, then it is continued holomorphically onto the whole domain.

Thus, $\hat{g}_k$ is holomorphic in $\hat{U}$. Let us consider the restriction of g_k on R^{n-1}. For each point of the boundary of the domain by repeating the previous arguments similarly to Lemma 3, it is easy to establish that g_k is continued holomorphically onto R^{n-1}.

Now let us prove that the number of surfaces is at most countable. In fact, for each surface $\mathcal{P}$ there exists the point λ such that only the finite number of surfaces passes through it.

Let $r_1, r_2, \ldots, r_{n+1}$ be rational numbers such that

$$\left|(r_1,\ldots,r_{n+1})-\lambda\right|<r_{n+1}$$

and, moreover, such that the other surfaces of $\sigma[A_1(\lambda)]$ do not pass through the neighbourhood

$$\left|\lambda-(r_1,\ldots,r_{n+1})\right|<r_{n+1}.$$

Thus, we get the one-to-one correspondence

$$\{\mathcal{P}_1,\ldots,\mathcal{P}_k\}\to(r_1,\ldots,r_{n+1}).$$

It means that the number of the surfaces is at most countable.

Lemma 4 is proved.

<u>Lemma 5.</u> The set $\sigma[A_1(\lambda)]\cap\ldots\cap\sigma[A_{n-1}(\lambda)]\cap R^n$ consists of at most countable number of the curves γ_m with the following properties:

1) $\gamma_m=\left\{\lambda:\lambda_k=\varphi_m^{(k)}(\lambda_1)\right\}$, where $\varphi_m^{(k)}$ is the analytic function.

2) These curves intersect at most countable number of the points and the intersection points do not accumulate in the finite part of R^n.

Proof. Let $\wp_1,\ldots,\wp_{n-1}$ be the spectrum-surfaces of the first, second, ..., and (n–1)-th problems, correspondingly

$$\wp_j=\left\{\lambda:\lambda_n=p_j(\lambda_1,\ldots,\lambda_{n-1})\right\};\quad j=1,2\ldots n-1.$$

Denote

$$\Phi_j(\lambda_1,\ldots,\lambda_{n-1})=\lambda_n-p_j(\lambda_1,\ldots,\lambda_{n-1});\quad j=1.2\ldots n-1.$$

The Jacobian of this system is

$$J=\begin{vmatrix}\dfrac{\partial\Phi_1}{\partial\lambda_2} & & \dfrac{\partial\Phi_1}{\partial\lambda_n}\\ \vdots & & \vdots\\ \dfrac{\partial\Phi_{n-1}}{\partial\lambda_2} & \cdots & \dfrac{\partial\Phi_{n-1}}{\partial\Phi_n}\end{vmatrix}.$$

If $\lambda_1=\overset{0}{\lambda}_1,\ldots,\overset{0}{\lambda}_{n-2}=\overset{0}{\lambda}_{n-2}$, then $\varphi(\lambda_{n-1})-p_1(\lambda_1^0,\ldots,\lambda_{n-2}^0,\lambda_{n-1})\equiv 0$ for some function $\lambda_{n-1}=\varphi(\lambda_{n-1})$.

It follows that

$$\frac{\partial\Phi_1}{\partial\lambda_{n-1}}=\frac{\partial\varphi_1}{\partial\lambda_{n-1}}=-\frac{\left(B_{1,n-1}u^1,u^1\right)}{\left(B_{1,n-1}u^1,u^1\right)},$$

where $u^1\in\operatorname{Ker}A_1(\lambda)$, $u^1\neq 0$ (see lemma 3).

For the other functions p_j and the points λ_j we have

$$J=\begin{vmatrix}\dfrac{\left(B_{12}u^1,u^1\right)}{\left(B_{1n}u^1,u^1\right)} & \cdots & \dfrac{\left(B_{1n-1}u^1,u^1\right)}{\left(B_{1n}u^1,u^1\right)} & 1\\ \vdots & \vdots & \vdots & \vdots\\ \dfrac{\left(B_{n-1,2}u^{n-1},u^{n-1}\right)}{\left(B_{n-1,n}u^{n-1},u^{n-1}\right)} & \cdots & \dfrac{\left(B_{n-1,n-1}u^{n-1},u^{n-1}\right)}{\left(B_{n-1,n}u^{n-1},u^{n-1}\right)} & 1\end{vmatrix}=$$

$$= \prod_{k=1}^{n-1} \left(B_{kn} u^k, u^k \right) \cdot \det \left(\left(B_{j\ell} u^j, u^j \right) \right), \text{ for all } j = 1, 2, \ldots, n-1, \ell = 2, 3, \ldots, n$$

According to the formulas (6) and (7) we obtain that $J \neq 0$.
Then in each small enough neighbourhood of the intersection point $\wp_1, \ldots, \wp_{n-1}$ the system of the equations

$$\left\{ \Phi_j = 0, \quad j = 1, 2, \ldots, n-1 \right\}$$

has the unique solution

$$\begin{cases} \lambda_2 = \varphi_1(\lambda_1) \\ \ldots \\ \lambda_n = \varphi_{n-1}(\lambda_1) \end{cases}$$

and for this curve points we have the formulas, like

$$\frac{\partial \lambda_n}{\partial \lambda_1} = -\left(\begin{vmatrix} B_{11} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,1} & \cdots & B_{n-1,n} \end{vmatrix} u^1 \otimes \ldots \otimes u^{n-1}, u^1 \otimes \ldots \otimes u^{n-1} \right) \cdot$$

(10).

$$\cdot \left(\begin{vmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{vmatrix} u^1 \otimes \ldots \otimes u^{n-1}, u^1 \otimes \ldots \otimes u^{n-1} \right)^{-1}$$

§3. The Approximation of the Joint Spectral Measure of the Multiparameter Problem

In this section we approximate the joint spectral measure of the self-adjoint operators $\overline{\Gamma}_j = \overline{\Delta_0^{-1}\Delta_j}, j = 1,\ldots,n$ by means of the spectral measures of operators $A_k(\lambda); k = 1,\ldots,n; \lambda \in R^n$.

Let σ be some analytic curve consisting of the points

$$\lambda \in \sigma[A_1(\lambda)] \cap \ldots \cap \sigma[A_n(\lambda)] \cap R^n,$$

and d be some arc in this curve such that d does not intersect other spectral curves; $d = \lambda^0\mu^0$, where λ^0 and μ^0 are the ends of the arc d, moreover, $\lambda^0 \notin d$, $\mu^0 \in d$ and $\lambda_1^0 < \mu_1^0$.

Let $\overline{\lambda}$ denote the midpoint of the line segment $\left[\lambda^0, \mu^0\right]$. Furthermore, let ν_j be the angle

$$\nu_j = \left(\Pr_{(\lambda_1,\lambda_j)}\left[\lambda^0,\mu^0\right]\right)^{\wedge}\overrightarrow{O\lambda_1},\ j = 2,\ldots,n, \text{ and}$$

$$B_\nu = tg\nu_n \cdot B_{nn} + \ldots + tg\nu_2 B_{n2} + B_{n1}.$$

And if we denote by ν_k^* the angle between the projection of the tangent to the curve σ at $\lambda^* \in d$ and the axis $O\lambda_1$, then from the results of the previous section (see (10)) it follows that

$$\operatorname{tg}\nu_k^* = (-1)^k \times$$

$$\times \left(\det \begin{pmatrix} B_{11} & \cdots B_{1,k-1} & B_{1,k+1} & \cdots & B_{1,n-1} \\ \vdots & & \vdots & & \vdots \\ B_{n-1,1} & \cdots B_{n-1,k-1} & B_{n-1,k+1} & \cdots & B_{n-1,n-1} \end{pmatrix} u^1 \otimes \ldots \otimes u^{n-1}, u^1 \otimes \ldots \otimes u^{n-1} \right) \times$$

$$\times \left(\det \begin{pmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{pmatrix} u^1 \otimes \ldots \otimes u^{n-2}, u^1 \otimes \ldots \otimes u^{n-1} \right)^{-1}, \qquad (11)$$

where $u^j \in \operatorname{Ker} A_j\left(\overset{*}{\lambda}\right)$, $j = 1, 2, \ldots, n-1$.

It is easy to prove, that for each $x^n \in H_n$ we have

$$\left(\left(\operatorname{tg}\nu_n^* B_{nn} + \ldots + \operatorname{tg}\nu_2^* B_{n2} + B_{n1} \right) x^n, x^n \right) =$$

$$= \left(\det \begin{pmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{pmatrix} u^1 \otimes \ldots \otimes u^{n-1}, u^1 \otimes \ldots \otimes u^{n-1} \right)^{-1} =$$

$$= \left(\Delta_0 u^1 \otimes \ldots \otimes u^{n-1} \otimes x^n, u^1 \otimes \ldots \otimes u^{n-1} \otimes x^n \right) \geq c \left\| u^1 \right\| \cdots \left\| u^{n-1} \right\| \cdot \left\| x^n \right\|; (c > 0).$$

It is clear that $\operatorname{tg}\nu_n \approx \operatorname{tg}\nu_n^*$ for small arcs d, consequently $B_j >> 0$.

Let $(H_n)_\nu$ be a Hilbert space consisting of H_n with the scalar product $(\cdot,\cdot)_\nu = \left(\cdot, B_\nu^* \cdot\right)$. We denote the orthogonal projection operators on the kernel of the operators $A_j^t\left(\lambda^*\right)$, $j=1,\dots,n-1$ considered as operators in $H_1\otimes\dots\otimes H_{n-1}\otimes\left(H_n\right)_\nu$ by $E^j_{\lambda^*}$.

We also denote by $E^n_\alpha(\lambda,\nu)$ the spectral family of the operator $\left[B_\nu^{-1}A_n(\lambda)\right]^t$ considered as the operator in $H_1\otimes\dots\otimes H_{n-1}\otimes\left(H_n\right)_\nu$. Furthermore, we set

$$\alpha_0=\frac{\mu_1^0-\lambda_1^0}{2}\begin{vmatrix}1 & tg\nu_2 & \cdots & tg\nu_{n-1}\\ tg\nu_n & 1 & \cdots & tg\nu_{n-2}\\ \cdots & \cdots & \cdots & \cdots\\ tg\nu_3 & tg\nu_4 & \cdots & 1\end{vmatrix}+\dots$$

$$+(-1)^{n-1}\frac{\mu_n^0-\lambda_n^0}{2}\begin{vmatrix}tg\nu_2 & tg\nu_3 & \cdots tg\nu_{n-1} & tg\nu_n\\ 1 & tg\nu_2 & \cdots tg\nu_{n-2} & tg\nu_{n-1}\\ \cdots & \cdots\cdots & & \cdots\\ tg\nu_4 & tg\nu_5 & \cdots 1 & tg\nu_2\end{vmatrix}=$$

$$=\begin{vmatrix}\frac{\mu_1^0-\lambda_1^0}{2} & \frac{\mu_2^0-\lambda_2^0}{2} & \cdots & \frac{\mu_n^0-\lambda_n^0}{2}\\ tg\nu_n & 1 & \cdots & tg\nu_{n-1}\\ \cdots & \cdots & \cdots & \cdots\\ tg\nu_2 & tg\nu_3 & \cdots & 1\end{vmatrix}.$$

Then

$$\alpha_0=\frac{\mu_1^0-\lambda_1^0}{2}\begin{vmatrix}1 & tg\nu_2 & \cdots & tg\nu_n\\ tg\nu_n & 1 & \cdots & tg\nu_{n-1}\\ \cdots & \cdots & \cdots & \cdots\\ tg\nu_2 & tg\nu_2 & \cdots & 1\end{vmatrix}.$$

According to the formula (11) we can see that $tg\nu_k^*$ becomes small enough if we multiply B_{j1} by a small enough number ε. Without changing the notations let us consider that the condition is satisfied, i. e., the absolute value of $tg\nu_k^*$ is small enough number. Then we have

$$\alpha_0 = \left(\mu_1^0 - \lambda_1^0\right)c,$$

where $c > 0$.
Furthermore, we set

$$E^n_{\left[\lambda^0,\mu^0\right]} = E^n_{\alpha_0}\left(\bar{\lambda}, \nu\right) - E_{-\alpha_0}\left(\bar{\lambda}, \nu\right)$$

(here $\bar{\lambda}$ is a midpoint of the line segment $\left[\lambda^0, \mu^0\right]$)
Let us also determine the operator

$$G_d = E^1_{\lambda^*} \cdots E^{n-1}_{\lambda^*} \cdot E^n_{\left[\lambda^0,\mu^0\right]} \tag{12}$$

acting on the space $H_1 \otimes \ldots \otimes H_{n-1} \otimes \left(H_n\right)_\nu$ Now we shall construct the projector equivalent to the previous one which projects onto the range of values of the operator G_d with respect to metrics $\langle \cdot, \cdot \rangle = \langle \cdot, \Delta_0 \cdot \rangle$.

Lemma 6. The projector in the space $\langle H \rangle$, which is equivalent to the projector G_d, can be represented in the form

$$\Phi_d = C^{-1}_{G_d} G_d \left(B^t_\nu\right)^{-1} \Delta_0,$$

where C_{G_d} is defined from the equation

$$\left(u, B^t_\nu C_{G_d} v\right) = \left(u, \Delta_0 v\right)$$

for arbitrary elements

$$u, v \in R\left(G_d\right).$$

To prove the lemma see [10], lemma 9.

We wish to show that for the small arcs d the operator Φ_d represents some suitable approximation of the operator

$$F_d = \int_d dE^1_{\alpha_1} \cdots E^n_{\alpha_n},$$

where E^j is the spectral family of the operator

$$\Gamma_j = \Delta_0^{-1} \Delta_j, \qquad j \in \{1, 2, \ldots n\}.$$

Let $d \subset \hat{d} \subset \sigma_m$, $\hat{d} \cap \sigma_{m'} = \emptyset$, $m' \neq m$ and $\hat{d}$ contain both ends and $\hat{J}$ be the parallelepiped which is parallel to the coordinate axis, moreover, $\hat{d} \in \hat{J}$. Now we set

$$\begin{matrix} f_{11}(\nu) = 1 & f_{12}(\nu) = tg\nu_2 & \cdots & f_{1n}(\nu) = tg\nu_n \\ f_{21}(\nu) = tg\nu_n & f_{22}(\nu) = 1 & \cdots & f_{2n}(\nu) = tg\nu_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1}(\nu) = tg\nu_2 & f_{n2}(\nu) = tg\nu_3 & \cdots & f_{nn}(\nu) = 1 \end{matrix}$$

Let $t_{jk}(\nu)$ be the cofactor of the element $f_{jk}(\nu)$ of the matrix $(f_{jk}(\nu))_{n \times n}$.

It is easy to see that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not joint eigenvalue of the operators $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_n$, then $\Gamma_1(\lambda, \nu)$ is invertible. In fact, if $\Gamma_1(\lambda, \nu)x = 0$, $x \neq 0$, then $F\{\lambda\} \neq 0$. Now it follows from [11] (see VI §5) that there exists $y \in \langle H \rangle$ such that $\Gamma_j y = \lambda_j y$, $j = 1, 2, \ldots, n$. But it is a contradiction.

Theorem 1. Let both ends $\overset{0}{\lambda}, \overset{0}{\mu}$ of the arc d not be joint eigenvalues of the operators $\overline{\Gamma}_1, \ldots \overline{\Gamma}_n$ and denote

$$W_d = \alpha_0^2 \left| \Gamma_1\left(\overset{0}{\lambda}, \nu\right) \cdot \Gamma_1\left(\overset{0}{\lambda}, \nu\right) \right|^{-1},$$

where α_0 is determined according to the formulas (11) and d is inside of a small enough arc $\hat{d}$.

Then for each $g \in \langle H \rangle$, $f \in D\left(W_d^{\gamma}\right)$ and for $0 < \gamma < \frac{1}{2}$ we have the estimation:

$$\left|\left\langle g,(F_d-\Phi_d)F_{\hat{j}}f\right\rangle\right| \le \alpha_0^q C\left(\gamma,\hat{j}\right)\Big\{ \langle\langle g\rangle\rangle \left\langle\left\langle W_d^\gamma F_d f\right\rangle\right\rangle +$$

$$+\langle\langle \Phi_d g\rangle\rangle \left[\langle\langle f\rangle\rangle + \left\langle\left\langle W_d^\gamma f\right\rangle\right\rangle\right]\Big\}, \qquad (15)$$

where the constant $C\left(\gamma,\hat{j}\right)$ does not depend on the location of d on $\hat{d}$ and $0<q<1$.

Corollary 1. If $f_n \to f$ and $W_d^\gamma f_n \to W_d^\gamma f$ for some $0<\gamma\le\frac{1}{2}$ and $f_n,\ f\in D\left(W_d^\gamma\right)$, then

$$(F_d-\Phi_d)F_{\hat{j}}f_n \xrightarrow{\ s\ } (F_d-\Phi_d)F_{\hat{j}}f .$$

Corollary 2. If the points $\xi^{(n)}, \eta^{(n)}$ are not joint eigenvalues of the operators $\overline{\Gamma}_1,\ldots\overline{\Gamma}_n$ and

$$\lim_{n\to\infty}\xi^{(n)} = \lim_{n\to\infty}\eta^{(n)},$$

then

$$\lim_{n\to\infty} F_{\left[\xi^{(n)},\eta^{(n)}\right]}F_{\hat{j}}f = \lim_{n\to\infty}\Phi_{\left[\xi^{(n)},\eta^{(n)}\right]}F_{\hat{j}}f$$

on the dense set

$$\bigcup_\gamma D\left(W_d^\gamma\right)\subset\langle H\rangle .$$

Proof. We set

$$\alpha' = \min_{\lambda'\in d}\left\{\left|(\lambda_1'-\lambda_1)t_{11}(\nu)+\cdots+(\lambda_n'-\lambda_n)t_{1n}(\nu)\right|\right\}$$

$$\alpha'' = \max_{\lambda' \in d} \left\{ \left| (\lambda_1' - \lambda_1) t_{11}(\nu) + \dots + (\lambda_n' - \lambda_n) t_{1n}(\nu) \right| \right\}.$$

It is clear that

$$\alpha' \langle\langle F_d f \rangle\rangle \le \langle\langle \Gamma_1(\lambda, \nu) F_d f \rangle\rangle \le \alpha'' \langle\langle F_d f \rangle\rangle. \tag{16}$$

Now, we shall represent the difference $(F_d - \Phi_d) F_{\hat{j}}$ as a sum of several terms and estimate all of them separately.

Let $\hat{\alpha}$ be a length of the arc $\hat{d}$ which is less than 1 and also $\alpha_0 \in (0,1)$.

We draw parallel hyperplanes through the line $\left[\lambda^0, \mu^0\right]$ with the distances between them equal to $\alpha_0^{1-\varepsilon_0}$ and denote their points of intersection with the curve $\sigma_m \supset d$ by

$$\lambda^k, \ \mu^k; \ k = 1, 2, \dots, r.$$

Let us choose the natural number r such that the following relation

$$(r-1)\alpha_0^{1-\varepsilon_0} < \alpha_0^{(1-\varepsilon_0)/2} \le r\alpha_0^{1-\varepsilon_0}$$

holds. If $\hat{\alpha}$ is chosen to be small enough then the whole arc $\lambda^r \mu^r$ can be put into the parallelepiped

$$\overset{\circ}{J} = \left\{ \lambda : a_1' < \lambda_1 < a_2', \dots, a_1^n < \lambda_n < a_2^n \right\},$$

such that the distance $\bar{\lambda}$ from the boundary of this parallelepiped is more than $c_1 \cdot \alpha_0^{(1-\varepsilon_0)/2}$ and the other points of the curve σ_m out of $\overset{\circ}{J}$.

Denote

$$d_k' = \lambda^{k-1} \lambda^k, \quad d_k'' = \lambda^{k-1} \lambda^k,$$

then

$$\left(F_d-\Phi_d\right)F_{\hat{j}}=\left(I-\Phi_d\right)F_d-\Phi_d\left(F_{\hat{j}}-F_{\underset{J}{\circ}}\right)-$$
$$\sum_{k=2}^{n}\Phi_d\left(F_{d'_k}+F_{d''_k}\right)-\Phi_d\left(F_{d'_k}+F_{d''_k}\right). \tag{17}$$

Let us estimate all four terms separately:
We have
1) $\Phi_d G_d = G_d$, therefore,

$$\left(I-\Phi_d\right)F_d=\left(I-\Phi_d\right)\left(I-G_d\right)F_d$$

and

$$\left\langle\left\langle\left(I-\Phi_d\right)F_d f\right\rangle\right\rangle\leq\sqrt{2}\left\langle\left\langle\left(I-G_d\right)F_d f\right\rangle\right\rangle\leq\sqrt{2}\left\langle\left\langle\left(I-E^{1}_{\lambda}{}_{*}+E^{1}_{\lambda}{}_{*}-\right.\right.\right.$$
$$\left.\left.\left.-E^{1}_{\lambda}{}_{*}E^{2}_{\lambda}{}_{*}+\cdots-E^{1}_{\lambda}{}_{*}\cdots E^{n-1}_{\lambda}{}_{*}+E^{1}_{\lambda}{}_{*}\cdots E^{n-1}_{\lambda}{}_{*}-E^{1}_{\lambda}{}_{*}\cdots E^{n-1}_{\lambda}{}_{*}E^{n}_{\left[\lambda^0,\mu^0\right]}\right)F_d f\right\rangle\right\rangle$$

$$\leq\sqrt{2}\left\langle\left\langle\left(I-E^{1}_{\lambda}{}_{*}\right)F_d f\right\rangle\right\rangle+\cdots+\sqrt{2}\left\langle\left\langle E^{1}_{\lambda}{}_{*}\cdots E^{n-1}_{\lambda}{}_{*}\left(I-E^{1}_{\left[\lambda^0,\mu^0\right]}\right)F_d f\right\rangle\right\rangle.$$

According to lemma 1

$$\left(I-E^{1}_{\lambda}{}_{*}\right)F_d f\left\{A^{t}_{j}\left(\lambda^{*}\right)\Big|_{R\left[A^{t}_{j}\left(\lambda^{*}\right)\right]}\right\}^{-1}$$

$$\left(I-E^{1}_{\lambda}{}_{*}\right)\cdot\sum_{k=1}^{n}B^{t}_{jk}(\nu)\overline{\Gamma_k}\left(\lambda^{*},\nu\right)F_d f \tag{18}$$

it is clear that

$$\left\langle\left\langle \bar{\Gamma}_k\left(\lambda^*,\nu\right)F_d f\right\rangle\right\rangle \le c_2\alpha_0\left\langle\left\langle F_d f\right\rangle\right\rangle,\ j=1,2,\ldots,n-1 \tag{19}$$

(for k=1 it follows immediately from (16),and for the other k it is true too due to the same arguments).
Hence

$$\left\langle\left\langle\left(I-E^j_{\lambda^*}\right)F_d f\right\rangle\right\rangle \le c_3\alpha_0\left\langle\left\langle F_d f\right\rangle\right\rangle,\quad j=1,2,\ldots,n-1, \tag{20}$$

Furthermore, according to lemma 1 we have

$$\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)\left[B_\nu^{-1}A_n\left(\bar{\lambda}\right)\right]^t F_d W_d f =$$

$$=\sum_{k=1}^{n}\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)\left(B_\nu^1\right)^{-1}B^t_{nk}(\nu)\cdot\bar{\Gamma}_k\left(\bar{\lambda},\nu\right)W_d F_d f$$

and, consequently,

$$\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)F_d W_d f-\left(\left[B_\nu^{-1}A_n\left(\bar{\lambda}\right)\right]^t\right)^{-1}\left(I-E_{\left[\lambda^0,\mu^0\right]}\right)F_d W_d\bar{\Gamma}_1\left(\bar{\lambda},\nu\right)F_d f=$$

$$=\sum_{k=2}^{n}\left\{\left[B_\nu^{-1}A_n\left(\bar{\lambda}\right)\right]^t\right\}^{-1}\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)\left(B_\nu^t\right)^{-1}B^t_{nk}(\nu)\cdot\bar{\Gamma}_k\left(\bar{\lambda},\nu\right)W_d F_d f \tag{21}$$

(by definition $B_{n1}(\nu)=B_\nu$).
By denoting the operators

$$X=\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)F_d W_d,$$

$$Q=\overline{\Gamma}_1\left(\overline{\lambda},\nu\right)F_d,\quad P=\left\{\left[B_\nu^{-1}A_n\left(\overline{\lambda}\right)\right]^t\right\}^{-1}\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right),$$

$$V=\sum_{k=2}^{n}\left\{\left[B_\nu^{-1}A_n(\lambda)\right]^t\right\}^{-1}\left(I-E^n_{\left[\lambda^0,\mu^0\right]}\right)\left(B_\nu^t\right)^{-1}B^t_{nk}(\nu)\cdot\Gamma_k(\lambda,\nu)W_dF_df$$

we obtain the following operator equation

$$X-PXQ=V. \qquad (22)$$

Recall the following proposition from [10]

Lemma 7 Let H_0 be a Hilbert space, B be the self-adjoint operator in H_0, where $B\gg 0$, H_B be a Hilbert space which can be obtained from H_0 by introducing the scalar product

$$(u,\upsilon)_B=(u,B\upsilon)_0,\quad u,\upsilon\in D(B).$$

Furthermore, let P be a bounded self-adjoint operator in H_B and Q be a bounded self-adjoint operator in H_0 and

$$\|P\|_B\le\gamma,\quad \|Q\|_0\le\gamma^{-1}$$

Let the inverse operator $W=\left[I-(\gamma Q)^2\right]^{-1}$ exist as (possibly unbounded) operator on the dense subset

$$D(W)\subset H_0.$$

Let us assume that V is defined everywhere in H_0 and if $\upsilon\in H_0$, then $V\upsilon\in H_B$ and

$$\|V\upsilon\|_B\le C_V\|\upsilon\|_0.$$

Then the operator equation

$$X-PXQ=V$$

has the solution X which is defined in

$$D(X)=\bigcup_{\gamma} D\left(W^{1+\gamma}\right), \quad \gamma \in \left(0, \tfrac{1}{2}\right]$$

and which can be represented as a sum of convergent series in the sense of the metric $\|u\|_B$

$$X\upsilon=\sum_{n=0}^{\infty} P^n V Q^n \upsilon, \quad \upsilon \in D(X)$$

$$X\upsilon \in H_B \quad \upsilon \in D(X)$$

and the estimation

$$\|X\upsilon\|_B \le C_\upsilon C(\gamma)\left\|W^{1+\gamma}\upsilon\right\|_0$$

holds for all

$$\upsilon \in D\left(W^{1+\gamma}\right), \quad 0<\gamma \le \tfrac{1}{2}.$$

We can see that the equation(22) satisfies the hypotheses of lemma 7 and $WF_d = W_d F_d$ (Here the main points are the inequalities

$$\left\langle\left\langle \overline{\Gamma}_j\left(\overset{*}{\lambda}, \nu\right) F_d f \right\rangle\right\rangle \le c_4 \alpha_0^2 \langle\langle F_d f\rangle\rangle, \quad j=1,2,\ldots,n$$

which follow from the fact that the integrand under the calculation of $\langle\langle \overline{\Gamma}_j(\lambda,\nu)F_d f\rangle\rangle$ is equal to the following expression

$$\left(\lambda_1' - \overset{0}{\mu}_1\right) t_{j1}(\nu) + \cdots + \left(\lambda_n' - \overset{0}{\mu}_n\right) t_{jn}(\nu).$$

But for small arcs d we have

$$\frac{\overset{0}{\lambda}_n - \overset{0}{\mu}_n}{\overset{0}{\lambda}_1 - \overset{0}{\mu}_1} = tg\nu_n + O_n(\alpha_0)$$

and hence,

$$\left(\lambda_1' - \overset{0}{\mu}_1\right)t_{j1}(\nu) + \cdots + \left(\lambda_n' - \overset{0}{\mu}_n\right)t_{jn}(\nu) = C_5 \begin{vmatrix} 1 & tg\nu_2 & \cdots & tg\nu_n \\ \vdots & \vdots & \vdots & \vdots \\ O_1(\alpha_o) & O_2(\alpha_o) & \cdots & O_n(\alpha_o) \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2 & tg\nu_3 & \cdots & 1 \end{vmatrix},$$

where $O_k(\alpha_0)$, $k \in \{1,2,\dots,n\}$ are elements of the j-th row of this matrix. Thus, according to lemma 7 we have

$$\|Xg\| \le C_6 C(\gamma)\alpha_0 \left\langle\left\langle W_d^{1+\gamma} F_d f \right\rangle\right\rangle,$$

$$g \in D(W_d F_d)^{1+\gamma}.$$

Hence we obtain

$$\left\| \left(I - E^n_{\left[\overset{0}{\lambda}, \overset{0}{\mu}\right]} \right) F_d (W_d F_d) f \right\| \le C_6 C(\gamma)\alpha_0 \left\langle\left\langle W_d^{\gamma} \left(W_d F_d f \right) \right\rangle\right\rangle.$$

The operator $W_d F_d$ has the inverse in the $R(F_d)$ and since the set $\{W_d F_d f,\ f \in D(W_d)\}$ is dense in $\{F_d \langle H \rangle\}^1$, we have

$$\left\langle\left\langle \left(I - E^n_{\left[\overset{0}{\lambda}, \overset{0}{\mu}\right]} \right) F_d g \right\rangle\right\rangle = \left\langle\left\langle \left(I - E^n_{\left[\overset{0}{\lambda}, \overset{0}{\mu}\right]} \right) F_d g_1 \right\rangle\right\rangle \le C_6 C(\gamma)\alpha_0 \left\langle\left\langle W_d^{\gamma} \left(W_d F_d f \right) \right\rangle\right\rangle,$$

for all $g = g_1 + g_2 \in D\left(W_d^{\gamma}\right)$, where $g_1 \in \overline{R(F_d)}$ and $g_2 \in \operatorname{Ker} F_d$.

As a final result we obtain

$$\langle\langle (I-\Phi_d)F_d g\rangle\rangle \le \alpha_0 C_7 C(\gamma)\left\langle\left\langle W_d' F_d f\right\rangle\right\rangle +$$

$$\sqrt{2}\cdot nC_3\alpha_0\langle\langle F_d f\rangle\rangle \le \alpha_0 C_8 C'(\gamma)\left[\left\langle\left\langle W_d' F_d f\right\rangle\right\rangle + \langle\langle F_d f\rangle\rangle\right].$$

2) There exist numbers $\lambda_1', \mu_1', \ldots, \lambda_n', \mu_n'$ such that

$$F_{\overset{\circ}{j}} = E_{\delta_1}^1 E_{\delta_2}^2 \cdots E_{\delta_n}^n,$$

where

$$E_{\delta_j}^j = E_{\mu_j'}^j - E_{\lambda_j'}^j, \quad j = 1, 2, \ldots, n.$$

Then

$$\Phi_d\left(F_{\hat{j}} - F_{\overset{\circ}{j}}\right) = \Phi_d\left(I - E_{\delta_1}^1\right)F_{\hat{j}} + \cdots + \Phi_d\left(I - E_{\delta_{n-1}}^{n-1}\right)E_{\delta_{n-2}}^{n-2}\cdots$$
$$\cdots E_{\delta_1}^1 \cdot F_{\hat{j}} + \Phi_d\left(I - E_{\delta_n}^n\right)E_{\delta_{n-1}}^{n-1}\cdots E_{\delta_1}^1 \cdot F_{\hat{j}}.$$

According to the definition of the operator $\Gamma_1(\lambda, \nu)$, we have

$$\Gamma_1(\bar{\lambda}, \upsilon)F_{\hat{j}} = \Delta_0^{-1}\begin{vmatrix} A_1(\bar{\lambda}) & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & & \vdots \\ A_n(\bar{\lambda}) & B_{n2} & \cdots & B_{nn}\end{vmatrix} F_{\hat{j}}.$$

Therefore,

$$\Phi_d\left(I - E_{\delta_1}^1\right)F_{\hat{j}} = C_{G_d}^{-1} G_d\left(B_\nu^t\right)^{-1} \Delta_1(\bar{\lambda})\Gamma_1^{-1}(\bar{\lambda}, 0)\left(I - E_{\delta_1}^1\right)F_{\hat{j}},$$

where $\Delta_1(\bar{\lambda})$ is obtained from Δ_1 if we take $A_j(\bar{\lambda})$ instead of A_j, $j = 1,2,\ldots,n$. Since the distance from a point $\bar{\lambda}$ to the boundary of the parallelepiped $\hat{J}$ is chosen more than a const. $\alpha_0^{(1-\varepsilon_0)/2}$, we have

$$\Gamma_1^{-1}(\bar{\lambda},0)\left(I - E_{\delta_1}^{1}\right)F_{\hat{j}} \le C_9\alpha_0^{-(1-\varepsilon_0)/2}.$$

Furthermore,

$$E_{\lambda^*}^{j} A_j^{t}(\bar{\lambda}) = E_{\lambda^*}^{j}\left[A_j^{t}\left(\lambda^*\right) + \sum_{k=1}^{n}\left(\lambda_k^* - \bar{\lambda}\right)B_{jk}^{t}\right].$$

Then

$$G_d A_j^{t}(\bar{\lambda}) \le C_{10}\alpha_0, \quad j = 1,2,\ldots,n-1,$$

$$\text{(note that } \left(\lambda_k^* - \bar{\lambda}\right) \le C'\alpha_0 \text{).}$$

It is clear that

$$G_d\left(B_\nu^{t}\right)A_n^{t}(\bar{\lambda}) \le C_{11}\alpha_0,$$

Then we have the relation

$$\left\langle\left\langle \Phi_d\left(I - E_{\delta_1}^{1}\right)F_{\hat{j}} f\right\rangle\right\rangle \le C_{12}\alpha_0^{1-(1-\varepsilon_0)/2}\langle\langle f\rangle\rangle = C_{12}\alpha_0^{(1+\varepsilon_0)/2}\langle\langle f\rangle\rangle.$$

Similarly, for the other terms in (26) we conclude

$$\left\langle\left\langle \Phi_d\left(F_{\hat{j}} - F_{\underset{J}{\circ}}\right)f\right\rangle\right\rangle \le C_{13}\alpha_0^{(1+\varepsilon_0)/2}\langle\langle f\rangle\rangle. \tag{27}$$

3)Let ν_j^k be an angle between the projection of the line $\left[\bar{\lambda},\lambda_k\right]$ on the plane $\left(\lambda_1,\lambda_j\right)$ and the axis $\overrightarrow{O\lambda_1}$, like so ν_j^{k*} also be an angle between the projection of the line $\left[\lambda^*,\lambda^k\right]$ on the plane $\left(\lambda_1,\lambda_j\right)$ and the axis $\overrightarrow{O\lambda_1}$, $k\geq 2$.

It is clear that

$$\nu_j^k-\nu_j^{k*}<C_{14}\alpha_0 \qquad j=2,3,\dots,n.$$

Then

$$\Phi_{d_k}F_{\alpha'_k}=C_{G_d}^{-1}G_d\left(B_\nu^t\right)^{-1}\Delta_0 Fd'_k=$$

$$=C_{G_d}^{-1}G_d\left(B_\nu^t\right)^{-1^{-1}}\cdot\left\|\begin{matrix}1 & tg\nu_2^k & \cdots & tg\nu_n^k\\ \vdots & \vdots & \vdots & \vdots\\ tg\nu_2^k & tg\nu_3^k & \cdots & 1\end{matrix}\right\|^{-1}\cdot\left|\begin{matrix}B_{11}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & \vdots & \vdots\\ B_{n1}\left(\nu^k\right) & \cdots & B_{nn}\left(\nu^k\right)\end{matrix}\right|F_{\alpha'_k}=$$

$$=C_{G_d}^{-1}G_d\left(B_\nu^t\right)^{-1}\left\|\begin{matrix}1 & tg\nu_2^{k^*} & \cdots & tg\nu_n^{k^*}\\ \vdots & \vdots & \vdots & \vdots\\ tg\nu_2^{k^*} & tg\nu_3^{k^*} & \cdots & 1\end{matrix}\right\|^{-1}\left|\begin{matrix}B_{11}\left(\nu^{k^*}\right) & B_{12}\left(\nu^{k^*}\right) & \cdots & B_{1n}\left(\nu^{k^*}\right)\\ \vdots & \vdots & & \vdots\\ 0 & B_{n2}\left(\nu^{k^*}\right) & \cdots & B_{nn}\left(\nu^{k^*}\right)\end{matrix}\right|F_{\alpha'_k}+$$

$$+C_{G_d}^{-1}G_d\left(B_\nu^t\right)^{-1}\cdot$$

$$\left\|\begin{matrix}1 & tg\nu_2^k & \cdots & tg\nu_n^k\\ \vdots & \vdots & & \vdots\\ tg\nu_2^k & tg\nu_3^k & \cdots & 1\end{matrix}\right\|^{-1}\left|\begin{matrix}B_{11}\left(\nu^k\right) & B_{12}\left(\nu^k\right) & \cdots & B_{1n}\nu^k\\ \vdots & & & \vdots\\ B_{n1}\left(\nu^k\right) & 0 & \cdots & 0\end{matrix}\right|F_{\alpha'_k}+\Omega, \qquad (28)$$

where Ω is the operator, for which we have

$$\langle\langle\Omega\rangle\rangle \le C_{15}\alpha_0\left\langle\left\langle F_{d_k^i} f\right\rangle\right\rangle.$$

Let us prove that the first term on the right hand side of (28) has an upper bound. From the first equation of the system (6) it follows that

$$\begin{vmatrix} A_1\left(\lambda^*\right) & B_{13}\left(\nu^{\ell*}\right) & \cdots & B_{1n}\left(\nu^{\ell*}\right) \\ \vdots & \vdots & & \vdots \\ A_{n-1}\left(\lambda^*\right) & B_{n-1,3}\left(\nu^{\ell*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell*}\right) \end{vmatrix}^t f =$$

$$= \begin{vmatrix} B_{11}\left(\nu^{\ell*}\right) & B_{13}\left(\nu^{\ell*}\right) & \cdots & B_{1n}\left(\nu^{\ell*}\right) \\ \vdots & \vdots & & \vdots \\ B_{n-1,1}\left(\nu^{\ell*}\right) & B_{n-1,3}\left(\nu^{\ell*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell*}\right) \end{vmatrix}^t \cdot$$

$$\cdot\overline{\Gamma}_1\left(\lambda^*,\nu^{\ell*}\right)f + \begin{vmatrix} B_{12}\left(\nu^{\ell*}\right) & B_{13}\left(\nu^{\ell*}\right) & \cdots & B_{1n}\left(\nu^{\ell*}\right) \\ \vdots & \vdots & & \vdots \\ B_{n-1,2}\left(\nu^{\ell*}\right) & B_{n-1,3}\left(\nu^{\ell*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell*}\right) \end{vmatrix}^t \overline{\Gamma}_2\left(\lambda^*,\nu^{\ell*}\right)f.$$

By multiplying this equation on the left by $G_d\left(B_\nu^t\right)^{-1}B_{nj}^t\left(\nu^{\ell*}\right)$, j=2,..,n

and taking $\overline{\Gamma}_1\left(\lambda^*,\nu^{\ell*}\right)F_{d'_\ell}f$ instead of f and also taking into account that

$G_dA\left(\lambda^*\right)=0$, j=2,..,n we have

$$G_d\left(B_\nu^t\right)^{-1} B_{nj}^t\left(\nu^{\ell*}\right)\begin{vmatrix} B_{11}\left(\nu^{\ell*}\right) & B_{13}\left(\nu^{\ell*}\right) & \cdots & B_{1n}\left(\nu^{\ell*}\right) \\ \vdots & & \vdots & \vdots \\ B_{n-1,1}\left(\nu^{\ell*}\right) & B_{n-1,3}\left(\nu^{\ell*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell*}\right)\end{vmatrix}^t F_{d'_\ell} =$$

$$= -G_d\left(B_\nu^t\right)^{-1} B_{nj}^t\left(\nu^{\ell*}\right)\cdot \tag{29}$$

$$\cdot\begin{vmatrix} B_{12}\left(\nu^{\ell*}\right) & B_{13}\left(\nu^{\ell*}\right) & \cdots & B_{1n}\left(\nu^{\ell*}\right) \\ & \vdots & \vdots & \vdots \\ B_{n-1,2}\left(\nu^{\ell*}\right) & B_{n-1,3}\left(\nu^{\ell*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell*}\right)\end{vmatrix}^t \cdot \overline{\Gamma}_2\left(\lambda^*, \nu^{\ell*}\right)\overline{\Gamma}_1\left(\lambda^*, \nu^{\ell*}\right)F_{d'_\ell}.$$

It is easy to prove that

$$\left\langle\left\langle \overline{\Gamma}_j\left(\lambda^*, \nu^{\ell*}\right)F_{d'_\ell}\right\rangle\right\rangle \le \text{const}\cdot(k+1)\left(\alpha_0^{1-\varepsilon_0}\right)^2, \; j=2,\ldots,n$$

and

$$\left\langle\left\langle \overline{\Gamma}_1\left(\lambda^*, \nu^{\ell*}\right)F_{d'_\ell}\right\rangle\right\rangle \le k\cdot\text{const}\cdot\alpha_0^{1-\varepsilon_0}.$$

Then

$$\left\langle\left\langle \overline{\Gamma}_j\left(\lambda^*, \nu^{\ell*}\right)\overline{\Gamma}_1^{-1}\left(\lambda^*, \nu^{\ell*}\right)F_{d'_\ell}\right\rangle\right\rangle \le \text{const}\cdot\alpha_0^{1-\varepsilon_0}$$

and

$$\left\langle\left\langle C_{G_d}^{-1} G_d\left(B_\nu^t\right)^{-1}\right.\right.$$

$$\begin{vmatrix} 1 & tg\nu_2^{k*} & \cdots & tg\nu_2^{k*} \\ \vdots & \vdots & & \vdots \\ tg\nu_2^{k*} & tg\nu_3^{k*} & \cdots & 1 \end{vmatrix}^{-1} \begin{vmatrix} B_{11}\left(\nu^{k*}\right) & B_{12}\left(\nu^{k*}\right) & \cdots & B_{1n}\left(\nu^{k*}\right) \\ \vdots & \vdots & & \vdots \\ 0 & B_{n2}\left(\nu^{k*}\right) & \cdots & B_{nn}\left(\nu^{k*}\right) \end{vmatrix} \cdot F_{d_k^t} \rangle\rangle \leq \quad (30)$$

$$\leq C_{16} \cdot \alpha_0^{1-\varepsilon_0} \langle\langle F_{d_k^t} f \rangle\rangle.$$

Now let us estimate the second term on the right-hand side of the equation(28). Again according to the formulas(6) we have

$$\left(B_\nu^t\right)^{-1} A_n^t\left(\bar{\lambda}\right)\bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right)F_{d_k^t} = \left(B_\nu^t\right)^{-1} B_{11}^t\left(\nu^k\right)F_{d_k^t} +$$

$$\left(B_\nu^t\right)^{-1} B_{n2}^t\left(\nu^k\right)\bar{\Gamma}_2\left(\bar{\lambda}, \nu^k\right)\bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right)F_{d_k^t} + \ldots$$

$$+\left(B_\nu^t\right)^{-1} B_{nn}^t\left(\nu^k\right)\bar{\Gamma}_n\left(\bar{\lambda}, \nu^k\right)\bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right)F_{d_k^t},$$

If we multiply the last equation on the left hand side by

$$G_d \det\left(B_{jm}\left(\nu^k\right)\right); j = 1,2,\ldots,n-1; \quad m = 1,2,\ldots,n,$$

we obtain

$$G_d\left(B_\nu^t\right)^{-1} A_n^t\left(\bar{\lambda}\right).$$

$$\begin{vmatrix} B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right) \end{vmatrix} \cdot \left(B_\nu^t\right)^{-1}\left(B_\nu - B_{n1}\left(\nu^k\right)\right)^{-1} \bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right)F_{d_k^t} =$$

$$=-G_d\left(B_\nu^t\right)^{-1}A_n^t(\bar\lambda)\cdot\begin{vmatrix}B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & \vdots & \vdots\\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right)\end{vmatrix}\cdot\left(B_\nu^t\right)^{-1}B_{n1}^t\left(\nu^k\right)\bar\Gamma_{1\bar\lambda}^{-1}\left(\nu^k\right)F_{d_k}+$$

$$+G_d\left(B_\nu^t\right)^{-1}B_{n1}^t\left(\nu^k\right)\cdot\begin{vmatrix}B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & & \vdots\\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right)\end{vmatrix}\cdot F_{d_k}+$$

$$\sum_{\ell=2}^{n}G_d\left(B_\nu^t\right)^{-1}B_{n\ell}^t\left(\nu^k\right)\cdot$$

$$\begin{vmatrix}B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & \vdots & \vdots\\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right)\end{vmatrix}\cdot\bar\Gamma_\ell\left(\bar\lambda,\nu^k\right)\bar\Gamma_1\left(\bar\lambda,\nu^k\right)F_{d_k}. \tag{31}$$

Let us denote

$$X=G_d\left(B_\nu^t\right)^{-1}B_{n1}^t\left(\nu^k\right)\cdot\begin{vmatrix}B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & & \vdots\\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right)\end{vmatrix}\cdot F_{d_k},$$

and

$$V=G_d\left(B_\nu^t\right)^{-1}A_n^t(\bar\lambda)\cdot$$

$$\begin{vmatrix}B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right)\\ \vdots & \vdots & \vdots\\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right)\end{vmatrix}\cdot\left(B_\nu^t\right)^{-1}\left(B_\nu-B_{n1}\left(\nu^k\right)\right)^t\bar\Gamma_1^{-1}\left(\bar\lambda,\nu^k\right)F_{d_k}-$$

$$-\sum_{\ell=2}^{n} G_d\left(B_\nu^t\right)^{-1} B_{n\ell}^t\left(\nu^k\right).$$

$$\begin{vmatrix} B_{12}\left(\nu^k\right) & \cdots & B_{1n}\left(\nu^k\right) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}\left(\nu^k\right) & \cdots & B_{n-1,n}\left(\nu^k\right) \end{vmatrix} \cdot \overline{\Gamma}_\ell\left(\overline{\lambda},\nu^k\right)\overline{\Gamma}_1\left(\overline{\lambda},\nu^k\right)F_{d_k},$$

then we have

$$X-\left(B_\nu^t\right)^{-1} A_n^t(\overline{\lambda})G_d X\overline{\Gamma}_1^{-1}\left(\overline{\lambda},\nu^k\right)F_{d_k}=V. \tag{32}$$

Let us prove that the equation(32)satisfies the hypotheses of the following lemma.

Lemma 8. **(see[20], lemma 8a).Suppose that the estimates $\|P\|_B \le \partial_p$, $\|Q\|_0 < \partial_Q$, $\partial_P\partial_Q < 1$ hold for the bounded operators P and Q in the spaces H_B and H_0, respectively. Let the operator V be defined everywhere in H_0, moreover, for $\upsilon \in H_0$ we have $V\upsilon \in H_B$ and**

$$\|V\upsilon\|_B \le C_V \cdot \|\upsilon\|_0.$$

Then there exists a unique bounded solution of the equation $X-PXQ=V$, for which we have the expansion $X=\sum_{n=0}^{\infty} P^n V Q^n$ and the estimate

$$\|X\upsilon\|_B \le \frac{C_v}{1-\partial_P\partial_Q}\cdot\|\upsilon\|_0, \quad \upsilon \in H_0.$$

We have already known the estimate for the operator

$$P=\left(B_\nu^t\right)^{-1} A_n^t(\overline{\lambda})G_d \quad ,$$

namely

$$\langle\langle P\rangle\rangle \le \alpha_0$$

And for $Q = \overline{\Gamma}_1^{-1}\left(\overline{\lambda}, v^k\right)F_{d_k^l}$ we have

$$\left\langle\left\langle \overline{\Gamma}_1\left(\overline{\lambda}, v^k\right)F_{d_k} f\right\rangle\right\rangle \ge \min_{\lambda' \in d_k^l}\left\{\left|\det\begin{pmatrix} \lambda_1' - \lambda_1 & \lambda_2' - \lambda_2 & \lambda_n' - \lambda_n \\ tgv_n^k & 1 & tgv_{n-1}^k \\ \dots & \dots & \dots \\ tgv_2^k & tgv_3^k & 1 \end{pmatrix}\right|\right\} \ge$$

$$\ge C_{17}\alpha_0^{1-\varepsilon_0}\cdot k \cdot \left\langle\left\langle F_{d_k^l} f\right\rangle\right\rangle.$$

For the small enough arc we get

$$\langle\langle Q\rangle\rangle \le \alpha_0^{-1+\varepsilon_0}\cdot kC_{17} < \alpha_0^{-1}.$$

Similarly, it is possible to prove that

$$\langle\langle Vf\rangle\rangle_v \le C_{18}\alpha_0^{1-\varepsilon_0}\cdot\left\langle\left\langle F_{d_k} f\right\rangle\right\rangle$$

(here $\langle\langle\cdot\rangle\rangle$ is the norm in $H_1 \otimes \ldots \otimes H_{n-1} \otimes (H_n)_v$).

Then, according to lemma 7 we conclude

$$\langle\langle Xf\rangle\rangle \le C_{19}\alpha_0^{1-\varepsilon_0}\cdot\left\langle\left\langle F_{d_k} f\right\rangle\right\rangle,$$

hence,

$$\left\langle\left\langle \Phi F_{d_k} f\right\rangle\right\rangle \le C_{20}\alpha_0^{1-\varepsilon_0}\cdot\left\langle\left\langle F_{d_k} f\right\rangle\right\rangle, \quad k = 2, \ldots r.$$

The similar inequality can be also proved for $\Phi_d F_{d_k^l} f$, so we have

$$\left\langle\left\langle\sum_{k=2}^{r}\Phi_d\left(F_{d_k'}-F_{d_k''}\right)f\right\rangle\right\rangle\leq C_{21}\alpha_0^{1-\varepsilon_0}\cdot\sum_{k=2}^{r}\left(\left\langle\left\langle F_{d_k'}f\right\rangle\right\rangle+\left\langle\left\langle F_{d_k''}f\right\rangle\right\rangle\right)\leq$$

$$C_{21}\alpha_0^{1-\varepsilon_0}\sqrt{2r}\cdot\left\langle\left\langle\sum_{k=2}^{r}\left(F_{d_k'}+F_{d_k''}\right)f\right\rangle\right\rangle\leq C_{22}\alpha_0^{(1-\varepsilon_0)\frac{3}{4}}\cdot\langle\langle f\rangle\rangle.$$

4) Let us expand $\Phi_d F_{d_k'}$ as in the case 3),but let now ν_j^{o*} be the angle between the projection of the line $\left[\lambda^0,\lambda^*\right]$ on the plane $(\lambda_1;\lambda_j)$ and the axis $\overrightarrow{O\lambda_1}$, $j=2,\ldots,n$.

Furthermore, instead of ν_j^1 we take ν_j , $j=2,\ldots,n$.

For the first term on the right hand side the formula similar to (30) is satisfied. For the second term, taking ν and $(I+W_d)F_1'$ instead of ν^k and $F_{d_k'}$, respectively (taking into account that $B_\nu=B_{n1}(\nu)$), we obtain the equation of the form

$$X+\left(B_\nu^t\right)^{-1}A_n^t(\bar{\lambda})G_d X\bar{\Gamma}_1^{-1}\left(\bar{\lambda},\nu^k\right)F_{d_k'}=V,$$

where

$$X=G_d\cdot\begin{vmatrix}B_{12}(\nu)&\cdots&B_{1n}(\nu)\\ \vdots&\vdots&\vdots\\ B_{n-1,2}(\nu)&\cdots&B_{n-1,n}(\nu)\end{vmatrix}\cdot(I+W_d)F_{d_1'},$$

$$V=-\sum_{\ell=2}^{n}G_d\left(B_\nu^t\right)^{-1}B_{n\ell}^t(\nu)\cdot$$

$$\begin{vmatrix}B_{12}(\nu)&\cdots&B_{1n}(\nu)\\ \vdots&\vdots&\vdots\\ B_{n-1,2}(\nu)&\cdots&B_{n-1,n}(\nu)\end{vmatrix}\cdot\bar{\Gamma}_\ell(\bar{\lambda},\nu)(I+W_d)\bar{\Gamma}_1^{-1}(\bar{\lambda},\nu)F_{d_1'}.$$

It is easy to verify that (the similar inequalities have been already verified several times)

$$\langle\langle Vf\rangle\rangle\leq C_{23}\alpha_0^{\frac{3}{4}(1-\varepsilon_0)}\cdot\left\langle\left\langle F_{d_1'}f\right\rangle\right\rangle.$$

Thus, all the hypotheses of lemma 7 are satisfied.

Since

$$WF_{d_1^{\prime}} = (I + W_d) F_{d_1^{\prime}},$$

according to lemma 7 we have

$$\langle\langle Xf \rangle\rangle \le \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{24} C(\gamma) \cdot \left\langle\left\langle (I + W_d)^{1+\gamma} F_{d_k^{\prime}} f \right\rangle\right\rangle$$

for all

$$f \in D(I + W_d)^{1+\gamma}.$$

Take $(I + W_d)^{-1} f$ instead of f and ν instead of ν_k for the second term in the last part of (28), we obtain

$$\left\langle\left\langle C_{G_d}^{-1} \begin{vmatrix} 1 & tg\nu_2^k & \cdots & tg\nu_2^k \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^k & tg\nu_3^k & \cdots & 1 \end{vmatrix}^{-1} \cdot X (I + W_d)^{-1} F_{d_k^{\prime}} f \right\rangle\right\rangle =$$

$$\left\langle\left\langle C_{G_d}^{-1} G_d \begin{vmatrix} 1 & \cdots & tg\nu_n \\ \vdots & \vdots & \vdots \\ tg\nu_2 & \cdots & 1 \end{vmatrix}^{-1} \cdot \begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu) \end{vmatrix} F_{d_k^{\prime}} f \right\rangle\right\rangle \le$$

$$\le \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{25} C(\gamma) \cdot \left[\langle\langle f \rangle\rangle + \left\langle\left\langle W_d^{\gamma} f \right\rangle\right\rangle \right]$$

for all $f \in D\left(W_d^{\gamma}\right)$.

Furthermore, for $F_{d_k^{\prime}}$ the similar inequality also holds and we obtain the relation

$$\left\langle\left\langle \Phi_d\left(F_{d_1'}+F_{d_1''}\right)f\right\rangle\right\rangle \le \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{26}C(\gamma)\cdot\left[\left\langle\left\langle W_d^{\gamma} f\right\rangle\right\rangle\right].$$

From the results of 1), 2), 3), 4) it follows that the formula (15) is true. The theorem 1 is proved completely.

§4. The Integral Representation of the Spectral Measures of the Multiparameter Problems

In this section applying the theorem 1 we represent the joint spectral family of the operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ with respect to $E^j_\lambda, \quad j = 1, 2, \ldots, n$.

Theorem 2. Let Δ_0 be a uniformly positive operator $\Delta_0 >> 0$, suppose that the relation (7) holds and $A_1, \ldots, A_{n-1}$ are self-adjoint operators with the discrete spectrum, A_n is an arbitrary self-adjoint operator.

Let $d = \overline{\overset{0}{\lambda}\overset{0}{\mu}}$ be the part of the arc of one of the curves of eigenvalues σ_p of the multiparameter operator system $(A_1(\lambda), \ldots, A_n(\lambda)), \quad \lambda \in R_n$, where $\overset{0}{\lambda} \notin d, \quad \overset{0}{\mu} \notin d$ and the closed convex hull does not contain the points of intersection with the other curves $\sigma_{p'}, \quad p' \neq p$.

Then there exists the sequence of the inscribed polygons of the arc d

$$\Phi_m = \bigcup_{k=1}^{r_n} \left[\lambda_{k-1,m} - \lambda_{k,m}\right] \quad , \overset{0}{\lambda} = \lambda_{0,m}, \quad \overset{0}{\mu} = \lambda_{r_m,m}$$

with the maximal length of the segments which tends to zero such that we have

$$F_d = s - \lim_{m \to \infty} \sum_{k=1}^{r_m} C^{-1} \left[E^1_{\lambda^*_{k,m}} \cdots E^{n-1}_{\lambda^*_{k,m}} E^n_{\left[\lambda_{k-1,m}\lambda_{k,m}\right]} \right],$$

$$\cdot E^1_{\lambda^*_{k,m}} \cdots E^{n-1}_{\lambda^*_{k,m}} E^n_{\left[\lambda_{k-1,m}\lambda_{k,m}\right]} \left(B^t_\nu\right)^{-1} \Delta_0$$

for any choice of the intermediate points $\lambda^*_{k,m}$ of the arc $\lambda_{k-1,m}\lambda_{k,m}$ and let $\nu^{k,m}$ be defined for the arc

$\lambda_{k-1,m}\lambda_{k,m}$, such as $\nu = (\nu_2, \ldots, \nu_n)$ is defined for the arc d.

The equality (36) can be rewritten in the integral form

$$F_d = \int_d C^{-1}_{E^1_\lambda \cdots E^{n-1}_\lambda E^n_{d\lambda}} \cdot E^1_\lambda \cdots E^{n-1}_\lambda E^n_{d\lambda} \left(B^t_{d\lambda} \right)^{-1} \Delta_0. \qquad (37).$$

Such a defined integral exists in the above mentioned sense and in the case when the arc d contains the points of the intersection with the other curves $\sigma_{p'}$ if for each point of that type we agree to consider

$E^1_\lambda(d) \cdots E^{n-1}_\lambda(d)$ instead of $E^1_\lambda \cdots E^n_\lambda$, where

$$E^j_\lambda(d) = \lim_{\lambda^{(m)} \to \lambda} E^{n-1}_{\lambda^{(m)}}, \quad \lambda^{(m)} \in d, \quad j = 1, \ldots, n-1.$$

For the spectral family $\mathbf{E}^j_\alpha$ of the operator $\Delta_0^{-1} \Delta_j$ the representation

$$\mathbf{E}^j_\alpha = \sum_{p=1}^{\infty} \int_{\substack{\sigma_p \\ \lambda_j < \delta_j \cdot \delta_n^{-1} \cdot \alpha}} C^{-1}_{E^1_\lambda \cdots E^{n-1}_\lambda E^n_{d\lambda}} \cdot E^1_\lambda \cdots E^{n-1}_\lambda E^n_{d\lambda} \left(B^t_{d\lambda} \right)^{-1} \Delta_0 \qquad (38)$$

holds.

In order to prove theorem 2 we apply the following Cordes lemmas (see [10] lemma 11, 12, 13).

Lemma 9. Let $\rho(\alpha)$ be the monotonically nondecreasing and continuous on the right function on the segment $[\alpha_1, \alpha_2]$. We set

$$\gamma(\alpha) = \operatorname*{Sup}_{\substack{\alpha_* \neq \alpha \\ \alpha_1 \le \alpha_* \le \alpha_2}} \frac{\rho(\alpha) - \rho(\alpha^*)}{\alpha - \alpha_*}$$

Then in each subinterval $\alpha_1' \le \alpha \le \alpha_2$, $\alpha_1 < \alpha_1' \le \alpha_2'$ there exists at least one point $\overline{\alpha}$, for which we have

$$\gamma(\overline{\alpha}) = 2 \frac{\rho(\alpha_2) - \rho(\alpha_1)}{\alpha_2 - \alpha_1}$$

Lemma 10. Let us introduce in a separable Hilbert space with the scalar product $(u,\upsilon)_0$ the sequence of scalar products and corresponding metrics $(u,\upsilon)_n$ such that

$$a(u,u)_0 \le (u,u)_n \le b(u,u)_0, \quad n = 1,2,\ldots$$

and

$$\left|(u,\upsilon)_n - (u,\upsilon)_0\right| \le \varepsilon_n \|u\|_0 \cdot \|\upsilon\|_0,$$

$$\lim_{n\to\infty} \varepsilon_n = 0$$

where $a > 0$ and $b > 0$ do not depend on n.

Let C_n be the operator self-adjoint in H with respect to the scalar product

$$(u,\upsilon)_n, \quad n = 1,2,\ldots$$

Assume that

$$\lim_{n\to\infty} (C_n - iI)^{-1} f = (C_0 - iI)^{-1} f; \quad f \in H$$

and if $\varphi \in \mathrm{Ker}(C_0 - \alpha_0 I)$ for some α_0, then $\varphi \in D(C_n)$ and

$$\|(C_n - \alpha_0 I)\varphi\|_n \le a_n \|\varphi\|_n,$$

where the sequence (a_n) does not depend on φ and $\lim\limits_{n\to\infty} a_n = 0$.

Then for the spectral families E_α^n of the operators C_n we have the relation

$$\lim_{n\to\infty}\left(E_{\alpha'_n}^n E_{\alpha''_n}^n\right) f = \left(E_{\alpha_0+0}^n - E_{\alpha_0-0}^n\right) f, \quad f \in H,$$

for each pair α'_n, α''_n such that

$$\lim_{n\to\infty} \alpha'_n = \lim_{n\to\infty} \alpha''_n = \alpha_0, \quad \alpha'_n < \alpha_0 < \alpha''_n$$

and

$$\lim_{n\to\infty} \frac{a_n}{\alpha'_n - \alpha_0} = \lim_{n\to\infty} \frac{a_n}{\alpha''_n - \alpha_0} = 0.$$

If the point α_0 (in particular) is not a pointwise eigenvalue of the operator C, then we have

$$\lim_{n\to\infty} E^n_{\alpha_n} f = E^n_{\alpha_0} f, \quad f \in H$$

for each $\{\alpha_n\}$, such that $\lim_{n\to\infty} \alpha_n = \alpha_0$.

Lemma 11. Assume that the relation

$$P_k P_\ell = \frac{1}{\gamma_k - \gamma_\ell} P_k \left(M^*_k N_\ell - N^*_k M_\ell \right) P_\ell$$

holds for the operators of the orthogonal projection P_k in the Hilbert space H, $k, \ell = 1, 2, \ldots, n, \quad k \neq \ell, \{\gamma_k\}_0^n \subset R$ and $M_k N_k$ are bounded operators for which

$$\gamma_0 < \gamma_1 < \ldots < \gamma_n, \qquad \|N_k P_k\| \le C',$$

$$\|M_k P_k\| \le C |\gamma_k - \gamma_{k-1}|,$$

where C, C' are positive constants.

Then for the operator $P = \sum_{k=1}^{n} P_k$ we have the estimate

$$0 \le P \le 1 + 2CC' \frac{a}{b} (4\pi + 2b),$$

where

$$a = \min_{k \in \{1,\ldots,n\}} |\gamma_k - \gamma_{k-1}|, \quad b = \max_{k \in \{1,\ldots,n\}} |\gamma_k - \gamma_{k-1}|.$$

The proof of lemma 9 is not so difficult. To prove lemma 10 first of all it is necessary to show that

$$\lim_{n\to\infty}(C_n - zI)^{-1}f = (C_0 - zI)^{-1}f, \quad f \in H,$$

if $Jm\,z \neq 0$. The main point of all the next arguments is the inequality

$$\left|\left(Q_n\left(I - E^n_{\delta_n}\right)u, \upsilon\right)\right| \le \frac{a_n}{\min\{|\alpha''_n - \alpha_0|, |\alpha'_n - \alpha_0|\}} \cdot \|u\|_n \cdot \|\upsilon\|_n,$$

where Q_n is the orthogonal projection on the kernel $C_0 - \alpha_0 I$, in a sense of $(u,\upsilon)_n$ and $E^n_{\delta_n} = E^n_{\alpha''_n} - E^n_{\alpha'_n}$.

To prove lemma 11 one should apply the number inequality

$$\left|\sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{n} \frac{x_k y_\ell}{\alpha_k - \alpha_\ell}\right|^2 \le \left(\frac{4\pi + 2b}{a}\right)^2 \sum_{k=1}^{n}|x_k|^2 \sum_{k=1}^{n}|y_k|^2,$$

where $0 < a \le \alpha_k - \alpha_{k-1} \le b$, $k = 1,2,\ldots,n$ for any choice of the complex numbers x_k and y_k.

The proof of theorem 2. There exist monotone non-decreasing continuous on the right base functions $\rho(\alpha)$ according to which for each $u, \upsilon \in \langle H \rangle$ the function

$$\psi(\alpha) = \left\langle u, E^1_\alpha \upsilon \right\rangle$$

is absolutely contiuous at each point of the interval $-\infty < \alpha < \infty$.

For example, the function

$$\rho(\alpha) = \sum_{m=1}^{\infty} \frac{1}{m^2}\left(\varphi_m, E^1_\alpha \varphi_m\right),$$

is one of them, where (φ_m) is the base of $\langle H \rangle$.

The space H' of all the elements $\upsilon \in \langle H \rangle$, for which the relation

$$\left| \frac{d\left\langle\left\langle E^1_\alpha \upsilon \right\rangle\right\rangle^2}{d\rho(\alpha)} \right| < C(\upsilon)$$

holds for some constant $C(\upsilon)$ is dense in $\langle H\rangle$ for all $\alpha \in(-\infty,\infty)$.

Indeed, for each linear combination $\upsilon = \sum_{n=1}^{N} a_n\varphi_n$ of the elements φ_n, $n = 1,2,\dots$ this relation always holds, if the constant $C(\upsilon)$ is

$$C(\upsilon) = N^2 \sum_{n=1}^{N} |a_n|^2 n^2 .$$

If $\lambda \in d$, then

$$F_{\overset{0}{\lambda}\lambda} = \left(E^1_{\lambda_1} - E^1_{\overset{0}{\lambda_1}}\right)\left(E^2_{\lambda_2} - E^2_{\overset{0}{\lambda_2}}\right)\dots\left(E^n_{\lambda_n} - E^n_{\overset{0}{\lambda_n}}\right) =$$

$$= E^1_{\lambda_1}\cdot F_{\overset{0}{\lambda}\lambda} = E^1_{\lambda_1}\cdot\left(F_d - F_{\lambda\overset{0}{\mu}}\right) = E^1_{\lambda_1} F_d$$

Now we want to find a subdivision of the arc d such that the division points will be outside of the set $\sigma_p^{jt}\left(\overline{\Gamma}_j\right)_{j=1}^n$ (apply theorem 1).

We denote by $\hat{d}$ the closed arc on σ_m containing d. This arc is continued to both sides and does not contain the points of intersection with $\sigma_{m'}$, $m' \neq m$.

We set $\hat{d} = \hat{\lambda}\hat{\mu}$, where $\hat{\lambda}_1 = \hat{\alpha}'$, $\hat{\mu}_1 = \hat{\alpha}''$. For simplicity we denote $\alpha' = \overset{0}{\lambda_1}$ and $\alpha'' = \overset{0}{\mu_1}$ and then we obtain

$$\hat{\alpha}' < \alpha' < \alpha'' < \hat{\alpha}''$$

Let $\hat{J}$ be defined in the same way as in the theorem 1. In order to choose the necessary sequence of the polygons Φ_m let us divide the interval

$$\alpha' \le \alpha \le \alpha'' + \tfrac{3}{2m} \ (\alpha'' - \alpha')$$

into 2m+3 equal parts.

$$\alpha'_{k,n} = \alpha' + \frac{k}{2m}\ (\alpha'' - \alpha'),\quad k = 0,1,\ldots,2m+3,$$

where m is supposed to be large enough and we apply lemma 9 for

$$\alpha_1 = \alpha'_{2k-2,m},\quad \alpha'_1 = \alpha'_{2k-1,m},\quad \alpha'_2 = \alpha'_{2k,m},$$
$$\alpha'_1 = \alpha'_{2k+1,m},\quad k = 1,2,\ldots,m+1.$$

Then there exist m+1 points $\alpha_{k,m}$, $k = 1,2,\ldots,m+1$ such that

$$\alpha' < \alpha'_{1,m} \le \alpha_{1,m} \le \alpha'_{2,m} < \alpha'_{3,m} \le \alpha_{2,m} \le \alpha'_{4,m} \le$$
$$\le \ldots \le \alpha'_{2m,m} = \alpha'' < \alpha'_{2m+1,m} \le \alpha_{m+1,m} \le$$
$$\le \alpha'_{2m+2,m} < \alpha'_{2m+3,m} < \hat{\alpha}'.$$

For the intervals

$$\frac{1}{2}\left(\alpha_{k-1,m} + \alpha_{k,m}\right) \le \alpha \le \frac{1}{2}\left(\alpha_{k,m} + \alpha_{k+1,m}\right)$$

for 1<k<m and

$$\alpha_{1,m} \le \alpha \le \frac{1}{2}\left(\alpha_{1,m} + \alpha_{2,m}\right),$$
$$\frac{1}{2}\left(\alpha_{m,m} + \alpha_{m+1,m}\right) \le \alpha \le \alpha_{m+1,m}$$

for k=1 and k=m, correspondingly, we have

$$\left.\frac{\Delta\rho}{\Delta\alpha}\right|_{\alpha_{k,m}} = \frac{\rho(\alpha) - \rho\left(\alpha_{k,m}\right)}{\alpha - \alpha_{k,m}} \le \frac{4m}{\alpha'' - \alpha'}\left[\rho\left(\alpha'_{2k+1,m}\right) - \rho\left(\alpha_{2k-1,m}\right)\right].$$

It means in particular that $\alpha_{k,m} \notin \operatorname{Ker} \overline{\Delta_0 \overset{-1}{\Delta_1}}$,
because otherwise there would exist the vector φ_{n_0} such that

$$\left(E^1\left\{\alpha_{k,n}\right\}\varphi_{n_0}, \varphi_{n_0}\right) \ne 0$$

and therefore, we have

$$\left.\frac{\Delta\rho}{\Delta\alpha}\right|_{\alpha_{k,m}} = \frac{\rho(\alpha_{k,m}) - \rho(\alpha_{k,m} - \varepsilon)}{\varepsilon} \geq \frac{1}{2}\left(E^1\{\alpha_{k,n}\}\varphi_{n_0}, \varphi_{n_0}\right)\frac{1}{\varepsilon} \to \infty$$

contradicting the formula (39).

Let $\lambda_{k,m} = (\alpha_{k,m}, \beta_{k,m}, \ldots, \gamma_{k,m})$, where $\beta_{k,m}, \ldots, \gamma_{k,m}$ is defined such that $\lambda_{k,m} \in d$, $k=1,2,\ldots,m+1$

Let us draw the polygon

$$\Phi_m = \overline{\overset{0}{\lambda}\lambda_{1,m}\ldots\lambda_{m-1,m}\lambda_{m,m}\overset{0}{\mu}},$$

$$d_{k,m} \overset{def}{=} \lambda_{k,m}\lambda_{k+1,m} \subset \sigma_p$$

$$k=1,2,\ldots,m,$$

with the help of intermediate points $\overset{*}{\lambda}_{k,m}$ we form the operators $\Phi_{d_{k,m}}$ and $d_m = \lambda_{1,m}\lambda_{m+1,m}$.

Then for $g \in \langle H \rangle$, $f \in H'$ according to theorem 1 we have

$$\left|\left\langle g, \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{k,m}}\right) F_{\hat{j}} f\right\rangle\right| = \left|\left\langle g, \sum_{k=1}^{m} \left(F_{d_{k,m}} - \Phi_{d_{k,m}}\right) F_{\hat{j}} f\right\rangle\right| \leq$$

$$\leq C(\hat{j}) m^{-\frac{3}{5}} \{\langle\langle g \rangle\rangle \sum_{k=1}^{m} \left\langle\left\langle W_{d_{k,m}}^{\frac{1}{4}} F_{d_{k,m}} f\right\rangle\right\rangle +$$

$$+ \sum_{k=1}^{m} \left\langle\left\langle \Phi_{d_{k,m}} g\right\rangle\right\rangle \left(\left\langle\left\langle F_{\hat{j}} f\right\rangle\right\rangle + \left\langle\left\langle W_{d_{k,m}}^{\frac{1}{4}} F_{\hat{j}} f\right\rangle\right\rangle\right)\} \qquad (40)$$

Now we shall find the upper bounds of all the terms of the right hand side of the inequality (40).

It is easy to verify that

$$\sum_{k=1}^{m}\left\langle\left\langle W_{d_{k,m}}^{\frac{1}{4}} F_{d_{k,m}} f\right\rangle\right\rangle \leq \sqrt{mc(f)}\cdot \text{const} \tag{41}$$

and

$$\left\langle\left\langle W_{d_{k,m}}^{\frac{1}{4}} F_j f\right\rangle\right\rangle \leq \sqrt{mc_2(f)}, \tag{42}$$

if only $f \in H'$.

And now applying lemma 10 we prove that

$$\sum_{k=1}^{m} \Phi_{d_{k,m}} \leq c_{27} I, \tag{43}$$

where c_{27} does not depend on the choice of m.

According to the definition of Φ_d we have

$$E^{j}_{\lambda^*_{k,m}} \Phi_{d_{k,m}} = E^{n}_{\left[\lambda_{k-1,m}\lambda_{k,m}\right]} \Phi_{d_{k,m}} = \Phi_{d_{k,m}}, \quad j=1,2,\ldots,n-1$$

so

$$\left\langle\left\langle \Delta_0^{-\frac{1}{2}} \begin{vmatrix} A_1(\lambda_{k,m}) & B_{13} & \cdots & B_{1n} \\ \vdots & & \vdots & \vdots \\ A_{n-1}(\lambda_{k,m}) & B_{n-1,3} & \cdots & B_{n-1,n} \end{vmatrix} \Phi_{d_{k,m}} \upsilon \right\rangle\right\rangle =$$

$$= \left\| \begin{vmatrix} A_1(\lambda_{k,m}) & B_{13} & \cdots & B_{1n} \\ \vdots & & \vdots & \vdots \\ A_{n-1}(\lambda_{k,m}) & B_{n-1,3} & \cdots & B_{n-1,n} \end{vmatrix} \Phi_{\alpha_{k,m}} \upsilon \right\| \leq$$

$$\leq C_{28}(\alpha_{k+1,m} - \alpha_{k,m}) \left\| \Phi_{\alpha_{k,m}} \upsilon \right\| \leq$$

$$\leq C_{29}(\alpha_{k+1,m} - \alpha_{k,m}) \langle\langle \Phi_{\alpha_{k,m}} \upsilon \rangle\rangle,$$

(take into account that $\left\| A_j(\lambda_{k,m}) E^j_{\lambda_{k,m}^*} \right\| \le (\alpha_{k+1,m} - \alpha_{k,m}) \cdot C$)

for $\upsilon \in \langle H \rangle$.

Furthermore,

$$\left\langle\left\langle \Delta_0^{-\frac{1}{2}} A_n^t(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \right\rangle\right\rangle = \left\| A_n^t(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \right\| =$$

$$\left\| B^t_{\nu^m} \left(B^t_{\nu^m} \right)^{-1} A_n^t(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \right\| \le$$

$$\le C_{30} \left\| \Phi_{d_{k,m}} \upsilon \right\| \le C_{31} \left\langle\left\langle \Phi_{d_{k,m}} \upsilon \right\rangle\right\rangle$$

Also we have

$$(\alpha_{k,m} - \alpha_{\ell,m}) \left\langle \Phi_{d_{k,m}} f, \Phi_{d_{\ell,m}} f \right\rangle = -\left(\Delta_1(\lambda_{k,m}) \Phi_{d_{k,m}} f, \Phi_{d_{\ell,m}} f \right) +$$

$$\left(\Delta_1(\lambda_{\ell,m}) \Phi_{d_{k,m}} f, \Phi_{d_{\ell,m}} g \right) = -\left(\Delta_0^{-\frac{1}{2}} \Delta_1(\lambda_{k,m}) \Phi_{d_{k,m}} f, \Delta_0^{-\frac{1}{2}} \Phi_{d_{\ell,m}} g \right) + \left(\Delta_0^{-\frac{1}{2}} \Phi_{d_{k,m}} f, \Delta_0^{-\frac{1}{2}} \Delta_1 \Phi_{d_{\ell,m}} g \right),$$

for all $f, g \in \langle H \rangle$.

Therefore, the relation

$$\Phi_{d_{k,m}} \cdot \Phi_{d_{\ell,m}} = \frac{1}{\alpha_{k,m} - \alpha_{\ell,m}} \Phi_{d_{k,m}} \left\{ \left[\Delta_0^{-\frac{1}{2}} \Phi_{d_{k,m}} \right]^{(*)} \cdot \right.$$

$$\left. \left[\Delta_0^{-\frac{1}{2}} \Delta_1(\lambda_{\ell,m}) \Phi_{d_{\ell,m}} \right] - \left[\Delta_0^{-\frac{1}{2}} \Delta_1(\lambda_{k,m}) \Phi_{d_{k,m}} \right]^{(*)} \left[\Delta_0^{-\frac{1}{2}} \Phi_{d_{k,m}} \right] \right\} \Phi_{d_{\ell,m}},$$

holds.

Then, in view of lemma 10, we have

$$\left\langle\left\langle \sum_{k=1}^{m} \Phi_{d_{\ell,m}} \right\rangle\right\rangle \le C_{27} \cdot I \tag{43}$$

Thus, because in view of (40),(41),(42) and (43) we obtain

$$\left|\left\langle g, \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{\ell,m}} \right) F_{\hat{J}} f \right\rangle\right| \le C'\left(\hat{J}, f\right) \cdot m^{-1/10} \left\{ \langle\langle g \rangle\rangle + \left\langle g, \sum_{k=1}^{m} \Phi_{d_{\ell,m}} g \right\rangle^{1/2} \right\} \le \tag{44}$$

$$\le C''\left(\hat{J}, f\right) \cdot m^{-1/10} \langle\langle g \rangle\rangle$$

Taking into account that g is arbitrary this estimate gives

$$\left\langle\left\langle \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{k,m}} \right) F_{\hat{J}} f \right\rangle\right\rangle \le C\left(\hat{J}, f\right) \cdot m^{-1/10} \tag{45}$$

for $f \in H'$.

Then prove this estimate for any $f \in \langle H \rangle$.

Let now $\hat{J}$ run a sequence $\hat{J}_m$, $m = 1, 2, \ldots$ such that $\bigcup_m \hat{J}_m = R^n$. We choose the orthonormal basis $\varphi_1, \varphi_2, \ldots$ used under defining the base function $\rho(\alpha)$ such that for each φ_α there exists m', such that the relation

$$F_{\hat{J}_m} \varphi_\ell = \varphi_\ell$$

holds.

It follows from (45) that

$$\left\langle\left\langle \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{k,m}} \right) F_{\hat{J}} f \right\rangle\right\rangle \le C_{30}(f) m^{-1/10}$$

for all $f \in \sum^{fin} \chi_k \varphi_k$. Taking into account that $\lim_{m\to\infty} F_{d_m} = F_d$, we obtain

$$\left(F_{d_m} - \lim_{m\to\infty}\sum_{k=1}^{m}\Phi_{d_{k,m}}\right)F_j f = 0$$

for all $f \in \sum^{\text{fin}} \chi_k \varphi_k$. Let now f be an arbitrary element from $\langle H\rangle$,

then for arbitrary $\varepsilon>0$ there exists $f' \in \sum^{\text{fin}} \chi_k \varphi_k$ such that

$$f=f'+f'',$$

where $\langle\langle f''\rangle\rangle \le \varepsilon$. Then

$$\left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f - F_d f \right\rangle\right\rangle \le \left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f - F_d f \right\rangle\right\rangle + \left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f'' \right\rangle\right\rangle \le$$

$$\le \left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f' - F_d f' \right\rangle\right\rangle + \langle\langle F_d f''\rangle\rangle + \left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f'' \right\rangle\right\rangle \le$$

$$\le \left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f' - F_d f' \right\rangle\right\rangle + \varepsilon\cdot\text{const},$$

hence,

$$\left\langle\left\langle \sum_{k=1}^{m}\Phi_{d_{k,m}} f - F_d f \right\rangle\right\rangle \le \varepsilon\cdot\text{const},$$

and

$$F_{d_m} f = \lim_{n\to\infty} \sum_{k=1}^{m}\Phi_{d_{k,m}} f, \quad f \in \langle H\rangle.$$

The following expression which corresponds to the polygon Φ_m

$$\Phi_{d_{0,m}} + \Phi_{\hat{d}_{m,m}} + \sum_{k=1}^{m} C^{-1}_{E^{1}_{\lambda^{*}_{k,m}} \cdots E^{n-1}_{\lambda^{*}_{k,m}} E^{n}_{\left[\lambda_{k-1,m}\lambda_{k,m}\right]}}$$

$$E^{1}_{\lambda^{*}_{k,m}} \cdots E^{n-1}_{\lambda^{*}_{k,m}} E^{n}_{\left[\lambda_{k-1,m}\lambda_{k,m}\right]} \left(B^{t}_{\nu k,m}\right)^{-1} \Delta_0$$

coincides with the following expression

$$\sum_{k=1}^{m} \Phi_{d_{k,m}} + \Phi_{d_{0,m}} + \Phi_{\tilde{d}_{m,m}} - \Phi_{d_{m,m}},$$

where

$$d_{0,m} = \lambda^{0} \lambda_{1,m}, \quad d_{m,m} = \lambda_{m,m}\mu^{0}.$$

Let us prove that $\Phi_{d_{0,m}} \xrightarrow{s} 0$ and $\Phi_{\tilde{d}_{m,m}} - \Phi_{d_{m,m}} \xrightarrow{s} 0$,

For $\Phi_{\tilde{d}_{m,m}}$ we have

$$\Phi_{\tilde{d}_{m,m}} = C^{-1}_{G_{\tilde{d}_{m,m}}} G_{d_{m,m}} \left(B^{t}_{\nu m}\right) \Delta_0,$$

where

$$G_{\tilde{d}_{m,m}} = E^{1}_{\lambda^{*}_{m,m}} \cdots E^{n-1}_{\lambda^{*}_{m,m}} E^{n}_{\left[\lambda_{m,m}\mu^{0}\right]}$$

and $E^{n}_{\left[\lambda_{m,m}\mu^{0}\right]}$ is the spectral family of the operator

$$\left[B^{-1}_{\nu m} A_n\left(\mu^{(m)}\right)\right]^{t} = \left\{B^{-1}_{\nu m} A_n\left(\mu^{0}\right) + \left(\mu^{0}_{1} - \mu_1\right) I\right\}^{t}$$

According to lemma 5 it follows that $E^n_{\left[\lambda_{m,m}\mu^0\right]} \to E^n_{\mu^0}$, where $E^n_{\mu^0}$-is the orthogonal projection on the kernel of the operator

$$\left[B_\nu^{-1}{}_m A_n\left(\mu^0\right)\right]^t$$

relating to $(\cdot,\cdot)_{\nu=\lim \nu^m}$.

Since

$$E^1_{\lambda^*_{m,m}} \cdots E^{n-1}_{\lambda^*_{m,m}} \to E^1_{\mu^0} \cdots E^{n-1}_{\mu^0},$$

we have

$$\lim_{m\to\infty} G_{\tilde{d}_{m,m}} f = E^1_{\mu^0} \cdots E^n_{\mu^0} f, \quad f \in H$$

It follows from the MPS theory that

$$\left(E^1_{\mu^0} \cdots E^n_{\mu^0}\right) H = F\left\{\mu^0\right\}\langle H\rangle.$$

According to lemma 5 we have

$$F\left\{\mu^0\right\} f = C^{-1}_{E^1_{\mu^0} \cdots E^n_{\mu^0}} \cdot E^1_{\mu^0} \cdots E^n_{\mu^0}\left(B^t_\nu\right)^{-1} \Delta_0.$$

Thus, the relation

$$\lim_{m\to\infty} \Phi_{\tilde{d}_{m,m}} f = F\left\{\mu^0\right\}$$

holds for all $f \in \langle H\rangle$.

We prove the similar equality for $\Phi_{d_{m,m}}$. Since $E^n_{[\lambda_{m,m},\lambda_{m+1,m}]}$ is the spectral family of the operator $\left[B^{-1}_{\nu^m} A_n\left(\mu^{(m)}\right)\right]^t$ where $\lambda_{m,m} = \mu^0$, the distance $\left|\mu^m \mu^0\right|$ with respect to the distance $\left|\mu^m \lambda_{m,m}\right|$ and $\left|\mu^m \lambda_{m+1,m}\right|$ tends to zero for $m \to \infty$. Thus, if $\lambda_{m,m} \neq \mu^0$ then for the operators $\left[B^{-1}_{\nu^m} A_n\left(\mu^m\right)\right]^t$ and $\left[B^{-1}_{\tilde{\nu}} A_n\left(\mu^0\right)\right]^t$, where $\nu = \lim_{m\to\infty} \nu_m$ the suppositions of the lemma 10 holds. Then we have

$$\lim_{m\to\infty} E^n_{[\lambda_{m,m},\lambda_{m+1,m}]} f = E^n_{\mu^0} f \quad ,f \in H$$

and, therefore,

$$\lim_{m\to\infty} \Phi_{d_{m,m}} f = F\left\{\mu^0\right\} f.$$

In the same way taking into account that $\lambda^0 \notin d$ for $\Phi_{d_{0,m}}$ we obtain $\lim_{m\to\infty} \Phi_{d_{0,m}} f = 0$.

This concludes the proof of the formula (36).
If the arc d_m contains the points of intersection with the other curves then the small arcs in the neighbourhood of this point λ can be neglected. Let d_λ be some small arc containing λ and $d_\lambda \subset d$.

Let $G_{d_\lambda} = E^1_{\lambda^*} \cdots E^n_{\lambda^*} E^n_{\xi\eta}$, where $\lambda^* \in d_\lambda$ and $d_\lambda = \xi\eta$ and let $|\lambda - \xi| = |\lambda - \eta|$. Then, applying lemma 10 we obtain

$$\lim_{d_\lambda \to \lambda} G_{d_\lambda} f = E^1_\lambda(d) \cdots E^{n-1}_\lambda(d) E^n_\lambda,$$

where E_λ^n is the operator of the orthogonal projection on the kernel of the operator $\left[B_{\tilde{\nu}}^{-1} A_n \begin{pmatrix} 0 \\ \mu \end{pmatrix}\right]^t$ with respect to the scalar product $\left(\cdot, B_\nu^t\right)$, $\tilde{\nu} = (\tilde{\nu}_2, \ldots, \tilde{\nu}_n)$, where $\bar{\nu}_j$ are the angles between corresponding tangents of the curve d at the point λ and their projections.

Thus,

$$F_{\sigma_m} = \int_{\sigma_m} C^{-1}_{E_\lambda^1 E_\lambda^2 \cdots E_\lambda^{n-1} E_{d_\lambda}^n} \cdot E_\lambda^1 \cdots E_\lambda^{n-1} E_{d\lambda}^{n-1} \left(B_{d\lambda}^t\right)^{-1} \Delta_0$$

The formula (38) follows from the facts like

$$E_\alpha^1 = F_{(-\infty,\alpha]\times(-\infty,\infty)\times(-\infty,+\infty)} = F_{(\alpha,\infty,\infty)}$$

Thus, theorem 2 is proved.

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