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INVERSE PROBLEMS OF IDENTIFYING THE TIME-DEPENDENT SOURCE COEFFICIENT FOR SUBELLIPTIC HEAT EQUATIONS

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Abstract. We discuss inverse problems of determining the time-dependent source coefficient for a general class of subelliptic heat equations. We show that a single data at an observation point guarantees the existence of a (smooth) solution pair for the inverse problem. Moreover, additional data at the observation point implies an explicit formula for the time-dependent source coefficient. We also explore an inverse problem with nonlocal additional data, which seems a new approach even in the Laplacian case.

1. Introduction.

Let $X = (X_1, \ldots, X_m)$ be a system of real smooth vector fields defined over a subset W of $\mathbb{R}^d$ satisfying the Hörmander condition (H): There exists a natural number r such that the vector fields $X_1, \ldots, X_m$ together with their r commutators span the tangent space at each point of W.

Let $\Omega \subset W$ be a bounded connected open subset with smooth boundary $\partial\Omega$ non-characteristic of $X = (X_1, \ldots, X_m)$. In Ω, we consider the following inverse problem of finding a pair (u, p) :

$$\begin{cases} \partial_t u(x,t) - \Delta_X u(x,t) = p(t)u(x,t) + f(x,t), & \text{in } \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

with additional condition $u(q,t) = w(t)$, $t \in [0,T]$, for a point $q \in \Omega$. Here

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$$\Delta_X := -\sum_{i=1}^{m} X_i^* X_i,$$

with $X_i^* = -X_i - \mathrm{div} X_i$, is a self-adjoint subelliptic operator (see [3]).

In the classical case when $X_i = \partial_{x_i},\ i = 1, \ldots, d$, one has the usual (elliptic) Laplacian Δ instead of the subelliptic operator Δ_X in (1) and there is a vast of literature on such parabolic inverse problems, especially, in 1D-cases. The existence and uniqueness of the solution for this inverse problem were initially established by Cannon, Lin, and Wang [7]. They utilized the method of time variable retardation along with a priori estimates. Additionally, for more general case, Prilepko and Soloviev [15] contributed by using a potential theoretic approach with the fundamental solution.

Furthermore, integral(nonlocal) overdetermination, expressed as $\int_\Omega u(x,t)\omega(x)dx = w(t)$, where ω and w are known functions and u is the solution of a specified parabolic equation, is also discussed in this paper. This approach can be instrumental in solving inverse coefficient problems. Notably, problems incorporating integral overdetermination in parabolic equations were first introduced and analyzed within the academic circle led by Prilepko [14], to the best of our knowledge. Subsequent in-depth investigations into these inverse problems were conducted by Cannon and Rundel [8], Ivanchov [11], Lesnic [13], among others, as referenced in their works.

There is a substantial body of literature on classical parabolic inverse problems, particularly in one-dimensional cases, as detailed in books [11] and [13] . Additional resources on integral overdetermination conditions can be found in [6] and [10] and the references cited therein. For a comprehensive analysis of related inverse problems in classical contexts, we also refer to references in [5] and [12].

The present paper aims to analyse inverse problems of recovering the time-dependent source parameter $p(t)$ in the Cauchy-Dirichlet problem for the subelliptic heat equation (1). First, in order to find a pair (u, p), we fix a point $q \in \Omega$ as an observation point for some time-dependent quantity. So, by using this additional date we recover the time-dependent source parameter $p(t)$. Interestingly, we discover that this method works well to study an inverse problem for the same model but with nonlocal additional data. The latter approach seems new even in the Laplacian case. Our proofs rely on subelliptic spectral theory arguments. Thus, we establish the existence and uniqueness for the inverse coefficient problem based on the eigenmodes (eigenvalues and eigenfunctions). It is well-known that the eigenmodes can be used to numerically solve these types of inverse problems (see, e.g. recent papers [2] and [4]).

Moreover, we also state that the solution to the inverse problem can be found explicitly in the case of an increase in the number of overdetermined data by potential theory techniques.

We organize our paper as follows. In Section 2, we prove the existence and uniqueness result with a single datum at an observation point by using a spectral theory approach. In Section 3, the time-dependent source parameter is found in an explicit form by applying the potential theory arguments. In Section 4, our technique from Section 2 is applied to treat a nonlocal case. Some interesting particular models are discussed in Section 5.

2. **Single datum.** Let Ω be a bounded connected open subset with smooth boundary $\partial\Omega$ non-characteristic of $X = (X_1, \ldots, X_m)$ and let $|H| > 0$, where the

set H is defined in (5). Consider the following inverse problem of finding a pair (u,p):

$$\begin{cases} \partial_t u(x,t) - \Delta_X u(x,t) = p(t)u(x,t) + f(x,t), & \text{in } \Omega \times (0,T), \\ u(x,0) = \varphi(x), \qquad x \in \Omega, \\ u(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2}$$

where $\varphi \in C^{2k}(\overline{\Omega})$ and $f \in C^{2k,0}(\overline{\Omega} \times [0,T])$ for integer $k > \frac{\tilde{\nu}}{4} + 1$. Here and after we understand $g \in C^1(\Omega)$ if $Xg \in C(\Omega)$.

The operator Δ_X is well-defined on $\{u \in H^1_{X,0}(\Omega) : \Delta_X u \in L^2(\Omega)\}$. Recall that $H^1_X(\Omega) = \{u \in L^2(\Omega) : X_i u \in L^2(\Omega), 1 \le i \le m\}$ and $H^1_{X,0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1_X(\Omega)$.

For any $x \in \bar{\Omega}$ and given $1 \le k \le r$, let $V_k(x)$ be the subspaces of the tangent space at x spanned by all commutators of $X_1, ..., X_m$ with length at most k. Note that the number r in the statement (H) in the introduction is called the Hörmander index. The Hausdorff dimension ν of Ω (or it can be also called the Métivier index of Ω) is defined as

$$\nu := \sum_{k=1}^{r} k\,(\nu_k - \nu_{k-1})$$

with $\nu_0 = 0$. Here it is assumed that for each $x \in \bar{\Omega}$, $\dim V_k(x)$ is a constant denoted by ν_k in a neighborhood of x.

Let us consider the following Dirichlet spectral problem of finding a nontrivial function ϕ and eigenvalue λ:

$$\begin{cases} -\Delta_X \phi(x) = \lambda \phi(x), & \text{in } \Omega, \\ \phi(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

By using the spectral theorem for compact self-adjoint operators, it can be shown that the eigenspaces are finite-dimensional and that the Dirichlet eigenvalues λ are real, positive, and have no limit point. The eigenspaces are orthogonal in the space of square-integrable functions and consist of smooth functions. In fact, the system of eigenfunctions $\phi_n(x)$, $n \in \mathbb{N}$, an orthonormal basis of $L_2(\Omega)$ (see, [3, Section 3.1]).

The eigenvalues can be arranged in increasing order:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to \infty,$$

where each eigenvalue is counted according to its geometric multiplicity. It is well-known that Weyl's asymptotic formula for the Dirichlet Laplacian ($\nu = d$) holds

$$\lambda_n \sim n^{\frac{2}{\nu}}. \tag{3}$$

The asymptotic formula (3) fails to hold for general Hörmander vector fields not satisfying the so-called Métivier condition. However, recently, in [3] for the operator Δ_X it was proved that Weyl's asymptotic formula

$$\lambda_n \sim n^{\frac{2}{\tilde{\nu}}} \tag{4}$$

holds if and only if $|H| > 0$. Here

$$H := \{x \in \Omega \mid \nu(x) = \tilde{\nu}\} \tag{5}$$

and $\nu(x) := \sum_{j=1}^{r} j\,(\nu_j(x) - \nu_{j-1}(x))$ (with $\nu_0(x) := 0$) is a pointwise homogeneous dimension and

$$\tilde{\nu} := \max_{x \in \bar{\Omega}} \nu(x). \tag{6}$$

Clearly, $\tilde{\nu} = \nu$ if $\dim V_k(x)$ is a constant in a neighborhood of x.

Let $q \in \Omega$ be a point such that $\phi_n(q)$, $n = 1, 2, ...$, is bounded with $\phi_n(q) \neq 0$ for some $n_0 \in \mathbb{N}$. Let $\mathbb{N}_q \subset \mathbb{N}$ be maximal set such that $\phi_n(q) \neq 0$ for all $n \in \mathbb{N}_q$. We use this point $q \in \Omega$ as an observation point for the quantity:

$$w(t) = u(q,t), \quad t \in [0,T]. \tag{7}$$

Set $\varphi_n = \int_\Omega \varphi(x)\phi_n(x)dx$ and $f_n(t) = \int_\Omega f(t,x)\phi_n(x)dx$.

Lemma 2.1. *If $\varphi \in C^{2k}(\bar{\Omega})$ and $\Delta_X^m \varphi = 0$, $m = 0, ..., k-1$, on $x \in \partial\Omega$ with $k > \frac{\tilde{\nu}}{4} + 1$, then we have*

$$\sum_{n=1}^{\infty} \lambda_n |\varphi_n| \leq c^{\frac{1}{2}} \left\|\Delta_X^k \varphi\right\|_{L_2},$$

where $c = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}$.

Proof. From Green's identities for the subelliptic operator Δ_X we get

$$\varphi_n = (-1)^k \frac{1}{\lambda_n^k} \left(\Delta_X^k \varphi\right)_n,$$

where $\left(\Delta_X^k \varphi\right)_n = \int_\Omega \Delta_X^k \varphi(x)\phi_n(x)dx$. By the Cauchy-Schwarz and Bessel inequalities we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n |\varphi_n| &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{k-1}} \left|\left(\Delta_X^k \varphi\right)_n\right| \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left|\left(\Delta_X^k \varphi\right)_n\right|^2\right)^{\frac{1}{2}} \\
&\leq c^{\frac{1}{2}} \left\|\Delta_X^k \varphi\right\|_{L_2}.
\end{aligned}$$

The series $c = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}$ is convergent by the Weyl-type asymptotic formula (4), since $k > \frac{\tilde{\nu}}{4} + 1$. □

Let us introduce:

$$D_k(\Omega) = \left\{\varphi \in C^{2k}(\bar{\Omega}) : \Delta_X^m \varphi = 0,\ m = 0, ..., k-1 \text{ on } \ x \in \partial\Omega\right\}$$

for some $k > \frac{\tilde{\nu}}{4} + 1$.

We have the following assumptions about the given functions.

1. $\varphi \in D_k(\Omega)$ with $\varphi_n \phi_n(q) \geq 0$ for $\forall n \in \mathbb{N}_q$ and $\varphi_{n_0}\phi_{n_0}(q) > 0$ for some $n_0 \in \mathbb{N}_q$;
2. $f \in C(\bar{\Omega} \times [0,T])$ and $f(x,t) \in D_k(\Omega)$ with $f_n(t)\phi_n(q) \geq 0$ for $\forall t \in [0,T]$ and for $\forall n \in \mathbb{N}_q$;
3. $w \in C^1[0,T]$ with $w(t) \neq 0$ for $\forall t \in [0,T]$ and $w(0) = \varphi(q)$.

The following theorem is valid for the existence and uniqueness of the inverse problem.

Theorem 2.2. *Assuming conditions (1)-(3) hold, a unique smooth pair (u,p), that is, $u \in C^{2k,1}(\bar{\Omega}, [0,T])$ and $p \in C[0,T]$, exists that solves the inverse problem (2) with (7).*

Proof of Theorem 2.2. The solution of (2) has the form:

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n e^{-\lambda_n t+\int_0^t p(\tau)d\tau}+\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)+\int_\tau^t p(s)ds}d\tau\right)\phi_n(x)$$

with $\varphi_n=\int_\Omega \varphi(x)\phi_n(x)dx$ and $f_n(t)=\int_\Omega f(t,x)\phi_n(x)dx$.

Let $r(t)=e^{-\int_0^t p(\tau)d\tau}$. So, we have

$$r(t)u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n e^{-\lambda_n t}+\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}r(\tau)d\tau\right)\phi_n(x).$$

From the additional condition (7) it follows that

$$r(t)u(q,t)=\sum_{n=1}^{\infty}\left(\varphi_n e^{-\lambda_n t}+\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}r(\tau)d\tau\right)\phi_n(q)=r(t)w(t)$$

or

$$r(t)=\frac{\sum_{n=1}^{\infty}\varphi_n e^{-\lambda_n t}\phi_n(q)}{w(t)}+\frac{1}{w(t)}\int_0^t\left(\sum_{n=1}^{\infty}\phi_n(q)f_n(\tau)e^{-\lambda_n(t-\tau)}\right)r(\tau)d\tau \tag{8}$$

in the case $w(t)\neq 0$.

From Green's identities for the subelliptic operator Δ_X we get

$$\varphi_n=\frac{(-1)^k}{\lambda_n^k}\int_\Omega \Delta_X^k\varphi(x)\phi_n(x)dx$$

for $\varphi\in C^{2k}(\bar{\Omega})$ and $\Delta_X^m\varphi=0$, $m=0,...,k-1$, on $x\in\partial\Omega$. We have

$$|\varphi_n|=\frac{1}{\lambda_n^k}\left|\int_\Omega \Delta_X^k\varphi(x)\phi_n(x)dx\right|.$$

Let $|\phi_n(q)|\leq M$ for some $M=const>0$. The majorant of the series

$$\sum_{n=1}^{\infty}\varphi_n e^{-\lambda_n t}\phi_n(q)$$

is $\sum_{n=1}^{\infty}|\varphi_n|$, that is, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{n=1}^{\infty}|\varphi_n|&=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^k}\left|\int_\Omega \Delta_X^k\varphi(x)\phi_n(x)dx\right|\\&\leq\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2k}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left|\int_\Omega \Delta_X^k\varphi(x)\phi_n(x)dx\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now we apply the Bessel inequality

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2k}}\sum_{n=1}^{\infty}\left|\int_\Omega \Delta_X^k\varphi(x)\phi_n(x)dx\right|^2\leq\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2k}}\left\|\Delta_X^k\varphi\right\|_{L_2}^2.$$

This series is convergent by the Weyl-type asymptotic formula (4).

The same inequality is true for the second series in (8). Then the second kind Volterra equation has a unique continuous solution in $[0,T]$ and it must be positive. Then

$$r(t)=e^{-\int_0^t p(\tau)d\tau}\Longrightarrow p(t)=-\frac{r'(t)}{r(t)}.$$

The function $r(t)$ is continuously differentiable if the series $\sum_{n=1}^{\infty}\varphi_n\lambda_n e^{-\lambda_n t}\phi_n(q)$ is uniformly convergent. It is the case since the majorant series $M\sum_{n=1}^{\infty}\lambda_n|\varphi_n|$ is convergent for $k>\frac{\tilde{\nu}}{4}+1$ by Lemma 2.1. ☐

Simple example. The inverse problem

$$\begin{cases} u_t = u_{xx} + p(t)u + (1+t)\sin x, \ 0<x<\pi; \ 0<t<T, \\ u|_{t=0} = \sin x, \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u(\frac{\pi}{2},t) = t+1, \end{cases}$$

satisfies all the assumptions of Theorem 2.2 in the one-dimensional case, since $\varphi(x)=\sin x$, $f(x,t)=(1+t)\sin x$, $q=\frac{\pi}{2}$ and $w(t)=t+1$. Consider the corresponding spectral problem

$$\begin{cases} -\phi''(x) = \lambda\phi(x), \\ \phi(0)=\phi(\pi)=0. \end{cases}$$

The eigenvalues are $\lambda_n = n^2, n=1,2,..$ and orthonormal eigenfunctions are $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n=1,2,....$

Because $\varphi\in D_2(0,\pi)$ and $f\in D_2(0,\pi)$ for $\forall t\in[0,T]$, and also $\varphi_n=(\varphi,\phi_n)=0$, $n>1$, $\varphi_1=\sqrt{\frac{\pi}{2}}$, $f_n(t)=(f,\phi_n)=0$, $n>1$, $f_1(t)=\sqrt{\frac{\pi}{2}}(t+1)$ and $\varphi_1\phi_1(\frac{\pi}{2})=1>0$, $f_1(t)\phi_1(\frac{\pi}{2})=1+t>0$ with $w(0)=1=\varphi(\frac{\pi}{2})$, all the assumptions of Theorem 2.2 are satisfied. This problem has the exact solution pair $u(x,t)=(1+t)\sin x$ and $p(t)=\frac{1}{1+t}$.

3. **Double data.** Let $\Omega\subset\mathbb{R}^d$ be bounded open set with piecewise smooth boundary $\partial\Omega$. Consider the inverse problem of identifying a pair (u,p):

$$\begin{cases} \partial_t u(x,t) - \Delta_X u(x,t) = p(t)u(x,t) + f(x,t), & \text{in } \Omega\times(0,T), \\ u(x,0)=\varphi(x), & x\in\Omega, \\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{9}$$

where $\varphi\in C_0^2(\Omega)$ and $f\in C_0^{2,0}(\Omega\times[0,T])$ are given functions. Assume that there exists $q\in\Omega$ such that $f(q,t)\in C^1[0,T]$ is continuously differentiable function with $f(q,t)\neq 0$. We now fix $q\in\Omega$ as an observation point for two time-dependent quantities:

$$w_1(t) := v_1(q,t), \quad w_2(t) := v_2(q,t), \quad t\in[0,T]. \tag{10}$$

Here

$$\begin{cases} \partial_t v_1(x,t) - \Delta_X v_1(x,t) = p(t)v_1(x,t) + f(x,t), & \text{in } \Omega\times(0,T), \\ v_1(x,0)=\varphi(x), & x\in\Omega, \\ v_1(x,t)=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{11}$$

and

$$\begin{cases} \partial_t v_2(x,t) - \Delta_X v_2(x,t) = p(t)v_2(x,t) + \Delta_X f(x,t), & \text{in } \Omega\times(0,T), \\ v_2(x,0)=\Delta_X\varphi(x), & x\in\Omega, \\ v_2(x,t)=0, & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{12}$$

Theorem 3.1. *Let $\varphi\in C_0^2(\Omega)$ and $f\in C_0^{2,0}(\Omega\times[0,T])$. Let $q\in\Omega$ be such that $f(q,t)\in C^1[0,T]$ is continuously differentiable function with $f(q,t)\not\equiv 0$. Let w_1 and w_2 defined in (10) be the observation data. Then there exists a unique smooth* *solution pair* (u, p)*, that is,* $u \in C^{2,1}(\bar{\Omega}, [0, T])$ *and* $p \in C[0, T]$*, for the inverse problem* (9) *with*

$$p(t)w_1(t) = w_1'(t) - w_2(t) - f(q, t).$$

Note that if p is found, then the solution u of the direct problem has the following representation

$$\begin{aligned} u(x,t) = &\exp\left(\int_0^t p(\tau)d\tau\right)\left[\int_\Omega h_D(x,y,t)\varphi(y)dy + \right. \\ &\left. + \int_0^t \int_\Omega h_D(x,y,t-\tau)\exp\left(-\int_0^\tau p(s)ds\right) f(y,\tau)dyd\tau\right], \end{aligned}$$

where h_D is the subelliptic heat kernel [3, Theorem 1.1] for the Cauchy-Dirichlet problem for the subelliptic heat equation in the cylindrical domain $\Omega \times [0, T)$.

Proof of Theorem 3.1. Let us recall the following transformation from [6]:

$$v(x,t) = r(t)u(x,t), \quad r(t) = \exp\left(-\int_0^t p(\tau)d\tau\right),$$

that is,

$$p(t) = -\frac{r'(t)}{r(t)}, \quad u(x,t) = \frac{v(x,t)}{r(t)}.$$

Thus, we have

$$\begin{cases} \partial_t v(x,t) - \Delta_X v(x,t) = r(t)f(x,t), & \text{in } \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{13}$$

We now fix a point $q \in \Omega$ as an observation point for two time dependent quantities:

$$\tilde{w_1}(t) := w_1(q,t), \quad \tilde{w_2}(t) := w_2(q,t), \quad t \in [0,T].$$

Here

$$\begin{cases} \partial_t w_1(x,t) - \Delta_X w_1(x,t) = r(t)f(x,t), & \text{in } \Omega \times (0,T), \\ w_1(x,0) = \varphi(x), & x \in \Omega, \\ w_1(x,t) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{14}$$

and

$$\begin{cases} \partial_t w_2(x,t) - \Delta_X w_2(x,t) = r(t)\Delta_X f(x,t), & \text{in } \Omega \times (0,T), \\ w_2(x,0) = \Delta_X\varphi(x), & x \in \Omega, \\ w_2(x,t) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{15}$$

It is known that the solutions of Cauchy-Dirichlet problems (14) and (15), correspondingly, can be presented by

$$w_1(x,t) = \int_0^t \int_\Omega h_D(x,y,t-\tau)r(\tau)f(y,\tau)\,dy + \int_\Omega h_D(x,y,t)\varphi(y)\,dy \tag{16}$$

and

$$w_2(x,t) = \int_0^t \int_\Omega h_D(x,y,t-\tau)r(\tau)\Delta_X f(y,\tau)\,dy + \int_\Omega h_D(x,y,t)\Delta_X\varphi(y)\,dy. \tag{17}$$

That is, for $q \in \Omega$ and for all $t \in [0, T]$ we have

$$\begin{aligned} w_1(q,t) = &\int_0^t \int_\Omega h_D(q,y,t-\tau) r(\tau) f(y,\tau)\, dy\, d\tau \\ &+ \int_\Omega h_D(q,y,t)\varphi(y)\, dy = \tilde{w}_1(t); \\ w_2(q,t) = &\int_0^t \int_\Omega h_D(q,y,t-\tau) r(\tau) \Delta_X f(y,\tau)\, dy\, d\tau \\ &+ \int_\Omega h_D(q,y,t)\Delta_X \varphi(y)\, dy = \tilde{w}_2(t). \end{aligned}$$

By differentiating $\tilde{w}_1$ and applying the Leibniz integral rule, then using $\partial_t h_D(x,y,t-\tau) = \Delta_X h_D(x,y,t-\tau),\ t > \tau,\ \partial_t h_D(x,y,t) = \Delta_X h_D(x,y,t),\ t > 0$, and Green's second identity for the operator Δ_X, we get

$$\begin{aligned} \tilde{w}_1'(t) = &\int_\Omega \partial_t h_D(q,y,t)\varphi(y)\, dy + r(t) f(q,t) \\ &+ \int_0^t \int_\Omega \partial_t h_D(q,y,t-\tau) r(\tau) f(y,\tau)\, dy\, d\tau \\ = &\int_0^t \int_\Omega \Delta_X h_D(q,y,t-\tau) r(\tau) f(y,\tau)\, dy\, d\tau \\ &+ r(t) f(q,t) + \int_\Omega \Delta_X h_D(q,y,t)\varphi(y)\, dy \\ = &\int_0^t \int_\Omega h_D(q,y,t-\tau) r(\tau) \Delta_X f(y,\tau)\, dy\, d\tau + r(t) f(q,t) \\ &+ \int_\Omega h_D(q,y,t)\Delta_X \varphi(y) dy \\ = &\tilde{w}_2(t) + r(t) f(q,t). \end{aligned} \tag{18}$$

Thus, we arrive at

$$\tilde{w}_2(t) = \tilde{w}_1'(t) - r(t) f(q,t). \tag{19}$$

Moreover, we have the following relations

$$\tilde{w}_1(t) := r(t) w_1(t), \quad \tilde{w}_2(t) := r(t) w_2(t), \quad t \in [0, T].$$

Plugging in (19) we obtain

$$p(t) w_1(t) = w_1'(t) - w_2(t) - f(q,t).$$

Also, the direct Cauchy-Dirichlet problems (13), (14), and (15) have a unique solution. It implies that there exists a unique classical pair (v, r), that is, (u, p) for the inverse problem (9). The proof is complete. □

4. **Nonlocal data.** Consider the following problem:

$$\begin{cases} \partial_t u(x,t) = \Delta_X u(x,t) + r(t) f(x,t), & \text{in } \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{20}$$

where $\varphi \in C^{2k}(\Omega)$ and $f \in C^{2k,0}(\Omega \times [0,T])$ for some integer $k > \frac{\tilde{\nu}}{4} + 1$.

Note that the equation (20) is equivalent to (2) (see the proof of Theorem 3.1). Now we consider the inverse problem of finding (u, r) from (20) with nonlocal datum

$$\int_{\Omega} \omega(x) u(x,t) dx = w(t), \quad t \in [0,T], \tag{21}$$

where $\omega(x) \in L_2(\Omega)$.

One could also discuss the case when (21) is replaced by the measurement in Section 2:

$$u(q,t) = w(t), \quad t \in [0,T],$$

where q is a given space observation point in Ω such that the sequence $\varphi_n(q)$, $n = 1, 2, \dots,$ is bounded. This is retrieved when one considers the Dirac delta distribution $\delta(x-q)$ centred at q in equation (21).

The following theorem is valid for the existence and uniqueness of the inverse problem.

Theorem 4.1. *Let the following conditions hold:*

1. $\varphi \in D_k(\Omega)$;
2. $f \in C(\Omega \times [0,T])$, $f(x,t) \in D_k(\Omega)$ *and* $\int_\Omega \omega(x) f(x,t)dx \neq 0$ *for* $\forall t \in [0,T]$;
3. $w \in C^1[0,T]$ *with* $w(t) \neq 0$ *for* $\forall t \in [0,T]$ *and* $w(0) = \int_\Omega \omega(x)\varphi(x)dx$.

Then there exists a unique smooth pair (u, r)*, that is,* $u \in C^{2k,1}(\bar{\Omega}, [0,T])$ *and* $r \in C[0,T]$*, for the inverse problem* (20)*-*(21)*.*

Proof of Theorem 4.1. The solution of (20) has the form:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} r(\tau) d\tau \right) \phi_n(x),$$

where $\varphi_n = \int_\Omega \varphi(x)\phi_n(x)dx$ and $f_n(t) = \int_\Omega f(x,t)\phi_n(x)dx$. So, we have

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-\lambda_n \varphi_n e^{-\lambda_n t} - \lambda_n \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} r(\tau) d\tau + f_n(t) r(t) \right) \phi_n(x).$$

From the over-determination condition (21) it follows that

$$\begin{aligned}
\int_\Omega \omega(x) u_t(x,t) dx = & \sum_{n=1}^{\infty} \Big(-\lambda_n \varphi_n e^{-\lambda_n t} \\
& -\lambda_n \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} r(\tau) d\tau \Big) \int_\Omega \omega(x)\phi_n(x)dx \\
& + r(t) \int_\Omega \omega(x) \sum_{n=1}^{\infty} f_n(t)\phi_n(x)dx \\
= & \omega'(t)
\end{aligned}$$

or

$$\begin{aligned}
r(t) = & \frac{w'(t) + \sum_{n=1}^{\infty} \lambda_n \varphi_n e^{-\lambda_n t} \int_\Omega \omega(x)\phi_n(x)dx}{\int_\Omega \omega(x) f(x,t)dx} + \frac{1}{\int_\Omega \omega(x) f(x,t)dx} \\
& \times \int_0^t \left(\sum_{n=1}^{\infty} \lambda_n f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \int_\Omega \omega(x)\phi_n(x)dx \right) r(\tau) d\tau
\end{aligned} \tag{22}$$

since $\int_\Omega \omega(x) \sum_{n=1}^{\infty} f_n(t)\phi_n(x)dx = \int_\Omega \omega(x) f(x,t)dx$.

Let $\int_\Omega \omega^2(x)dx \le M$ in (21) for some $M = const > 0$. The series

$$\sum_{n=1}^{\infty} \lambda_n \varphi_n e^{-\lambda_n t} \int_\Omega \omega(x)\phi_n(x)dx$$

is uniformly convergent since the majorant series $M\sum_{n=1}^{\infty} \lambda_n |\varphi_n|$ is convergent by Lemma 2.1. Thus, the Volterra integral equation (22) has a unique continuous solution. □

5. **Particular cases and conclusion. Laplacian**. In the classical case, when $X_i = \partial_{x_i}$, $i = 1, \dots, d$, one has the usual (elliptic) Laplacian Δ instead of the subelliptic operator Δ_X in (1). Our approach seems new even in this classical case.

Sub-Laplacians. Let $\mathbb{G} = (\mathbb{R}^d, \circ)$ be a stratified Lie group and $\{X_1, \dots, X_m\}$, $m \le d$, be a system of generators for $\mathbb{G}$, i.e. a basis for the first strata of $\mathbb{G}$. It satisfies the Hörmander condition (H). The sum of squares operator

$$\Delta_{\mathbb{G}} = -\sum_{i=1}^{m} X_i^2$$

is called the sub-Laplacian on $\mathbb{G}$. $\Delta_{\mathbb{G}}$ is elliptic if and only if $\mathbb{G}$ is same as $(\mathbb{R}^d, +)$. It is clear that the above results are valid on stratified Lie groups. Note that the class of stratified Lie groups includes the groups of Iwasawa type and the H-type groups. A simple example in $\mathbb{R}^3$ is the operator

$$(\partial_y + 2x\partial_z)^2 + (\partial_x - 2y\partial_z)^2.$$

Baouendi-Grushin operator. Let $z := (x, y) := (x_1, ..., x_m, y_1, ..., y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $m, k \ge 1$ and $m + k = d$. Let us consider the vector fields

$$X_i = \frac{\partial}{\partial x_i},\ i = 1, ..., m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j},\ \gamma \ge 0,\ j = 1, ..., k.$$

The Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ is defined by

$$\Delta_\gamma = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \triangle_x + |x|^{2\gamma}\triangle_y, \tag{23}$$

where $\triangle_x$ and $\triangle_y$ stand for the standard Laplacians in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively. Recall that the Baouendi-Grushin operator for a positive even integer γ is a sum of squares of vector fields satisfying the Hörmander condition (H). In that case, the results of the present paper hold for the Baouendi-Grushin operator (cf. [1]). A simple example in $\mathbb{R}^2$ is the following operator

$$\partial_{xx}^2 + x^2\partial_{yy}^2.$$

Conclusion. In conclusion, the findings presented in this paper demonstrate that the subelliptic operator under consideration encompasses a broad spectrum of cases, underlining its applicability in various scenarios. The adoption of the spectral theory approach in our analysis has shown its effectiveness, particularly due to its minimal reliance on additional data, which enhances its practicality. However, a significant challenge encountered in this methodology is the verification of the boundedness of the sequence of normalized eigenfunctions at a specific observation point q. This aspect is straightforward in the context of two-dimensional canonical domains for the Dirichlet Laplacian, for example. Nevertheless, this may not hold true in more complex geometrical configurations, where an appropriate observation point q might not exist, as elaborated in [9, Theorem 4].

On the flip side, our exploration of the potential theory approach has yielded interesting results, particularly in providing explicit solution pairs. This is indeed interesting, considering the rarity of such outcomes in the inverse problem theory. However, it's important to acknowledge the limitations of this approach, primarily its dependency on acquiring additional data at the observation point. This requirement could potentially restrict its applicability in certain scenarios.

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