Eur. Phys. J. C (2023) 83:1007
https://doi.org/10.1140/epjc/s10052-023-12086-6

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Bailey pairs for the q-hypergeometric integral pentagon identity

Ilmar Gahramanov[1,2,3,a], **Osman Erkan Kaluc**[1,b]

[1] Department of Physics, Bogazici University, Bebek, 34342 Istanbul, Turkey
[2] Institute of Radiation Problems, Azerbaijan National Academy of Sciences, B.Vahabzade St. 9, AZ1143 Baku, Azerbaijan
[3] Department of Mathematics, Khazar University, Mehseti St. 41, AZ1096 Baku, Azerbaijan

Received: 18 June 2023 / Accepted: 27 September 2023

Abstract In this work, we construct new Bailey pairs for the integral pentagon identity in terms of q-hypergeometric functions. The pentagon identity considered here represents the equality of the partition functions of certain three-dimensional supersymmetric dual theories. It can be also interpreted as the star-triangle relation for the Ising-type integrable lattice model.

Contents

1 Introduction

Bailey's lemma [1,2] is a powerful tool to derive hypergeometric identities (ordinary, trigonometric, and elliptic type). In this work, we construct new integral Bailey pairs for the pentagon identity in terms of q-hypergeometric functions. The pentagon identity can be interpreted as a Pachner's 3-2 move for triangulated three-dimensional manifolds. Such identities also play a role in the study of supersymmetric gauge theories, integrable models, knot theory, etc.[1]

Let $q, z \in \mathbb{Z}$ with $|q| < 1$. We define the infinite q-product

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1-zq^k). \tag{1.1}$$

We also adopt the following convention

$$(a,b;q)_\infty := (a;q)_\infty (b;q)_\infty. \tag{1.2}$$

Theorem 1.1 *Let $a_1, a_2, a_3, b_1, b_2, b_3, q \in \mathbb{C}$ and integers $m_i, n_i \in \mathbb{Z}$. Then*

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \frac{dz}{2\pi i z} (-q^{\frac{1}{2}})^{\sum_{i=1}^{3} \frac{|m_i+m|}{2} + \frac{|n_i-m|}{2}} z^{-\sum_{i=1}^{3}(\frac{|m_i+m|}{2} - \frac{|n_i-m|}{2})} \times \prod_{i=1}^{3} a_i^{-\frac{|m_i+m|}{2}} b_i^{-\frac{|n_i-m|}{2}} \frac{(q^{1+\frac{|m_i+m|}{2}}\frac{1}{a_i z}, q^{1+\frac{|n_i-m|}{2}}\frac{z}{b_i};q)_\infty}{(q^{\frac{|m_i+m|}{2}}a_i z, q^{\frac{|n_i-m|}{2}}\frac{b_i}{z};q)_\infty}$$
$$= (-q^{\frac{1}{2}})^{\sum_{i,j=1}^{3}\frac{|m_i+n_j|}{2}} \prod_{i,j=1}^{3} (a_i b_j)^{-\frac{|m_i+n_j|}{2}} \times \frac{(q^{1+\frac{|m_i+n_j|}{2}}\frac{1}{a_i b_j};q)_\infty}{(q^{\frac{|m_i+n_j|}{2}}a_i b_j;q)_\infty}, \tag{1.3}$$

where the balancing conditions are

$$\prod_{i=1}^{3} a_i b_i = q, \tag{1.4}$$

$$\sum_{i=1}^{3} m_i + n_i = 0, \tag{1.5}$$

and $\mathbb{T}$ represents the positively oriented unit circle.

We would like to mention that the integral identity represents the supersymmetric duality for three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories with the flavor symmetry[2] $SU(3)\times SU(3)\times U(1)$. This identity can also be writ-

[1] See some recent works [3–11].

[2] In this case parameters a_i and b_i stand for the flavor symmetry and z is the fugacity for the $U(1)$ gauge group.

[a] e-mail: ilmar.gahramanov@boun.edu.tr (corresponding author)
[b] e-mail: erkan.kaluc@boun.edu.tr

Published online: 07 November 2023

ten as the star-triangle relation[3] for some integrable model of statistical mechanics.

The proof of the form above is given in [8] for the balancing conditions[4]

$$\prod_{i=0}^{3} a_i = \prod_{i=0}^{3} b_i = q^{\frac{1}{2}}, \quad \sum_{i=0}^{3} m_i = \sum_{i=0}^{3} n_i = 0. \tag{1.6}$$

The absolute values can be eliminated by the identity [12]

$$\frac{(q^{1+\frac{|m|}{2}}/z;q)_\infty}{(q^{\frac{|m|}{2}}z;q)_\infty} = (-q^{-\frac{1}{2}}z)^{\frac{|m|-m}{2}} \frac{(q^{1+\frac{m}{2}}/z;q)_\infty}{(q^{\frac{m}{2}}z;q)_\infty}, \tag{1.7}$$

and one ends up with the following q-hypergeometric sum/integral identity [6–8]

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \prod_{i=1}^{3} \frac{(q^{1+\frac{m+m_i}{2}}\frac{1}{a_i z}, q^{1+\frac{n_i-m}{2}}\frac{z}{b_i};q)_\infty}{(q^{\frac{m+m_i}{2}}a_i z, q^{\frac{n_i-m}{2}}\frac{b_i}{z};q)_\infty} \frac{1}{z^{3m}} \frac{dz}{2\pi i z}$$
$$= \frac{1}{\prod_{i=1}^{3} a_i^{m_i} b_i^{n_i}} \prod_{i,j=1}^{3} \frac{(q^{1+\frac{m_i+n_j}{2}}\frac{1}{a_i b_j};q)_\infty}{(q^{\frac{m_i+n_j}{2}}a_i b_j;q)_\infty}. \tag{1.8}$$

2 Integral pentagon identity

In [6–8] it was shown that the identity (1.3) can be written as an integral pentagon identity

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \frac{dz}{2\pi i z} \prod_{i=1}^{3} \mathcal{B}[a_i, n_i+m; b_i z^{-1}, m_i - m]$$
$$= \mathcal{B}[a_1 b_2, n_1+m_2; a_3 b_1; n_3+m_1]$$
$$\times \mathcal{B}[a_2 b_1, n_2+m_1; a_3 b_2, n_3+m_2], \tag{2.1}$$

where we define the following function as

$$\mathcal{B}_m[a,n;b,m] = (-q^{\frac{1}{2}})^{\frac{|n|}{2}+\frac{|m|}{2}+\frac{|n+m|}{2}} a^{-\frac{|n|}{2}} b^{-\frac{|m|}{2}} (ab)^{\frac{|n+m|}{2}}$$
$$\times \frac{(q^{1+\frac{|n|}{2}}a^{-1}, q^{1+\frac{|m|}{2}}b^{-1}, q^{\frac{|n+m|}{2}}ab;q)_\infty}{(q^{\frac{|n|}{2}}a, q^{\frac{|m|}{2}}b, q^{1+\frac{|n+m|}{2}}(ab)^{-1};q)_\infty}. \tag{2.2}$$

In a general sense, any algebraic relation for operators $\mathcal{B}$

$$\mathcal{B}\mathcal{B}\mathcal{B} = \mathcal{B}\mathcal{B} \tag{2.3}$$

which can be interpreted as a 2–3 Pachner move of a triangulated three-dimensional manifold is called a pentagon relation [4,5]. Note that the integral pentagon identity (2.1) for the $\mathcal{N}=2$ supersymmetric $S^2\times S^1$ partition functions is supposed to be related to some topological invariant of corresponding 3-manifold via $3d$–$3d$ correspondence [12,13] that connects three-dimensional $\mathcal{N}=2$ supersymmetric theories and triangulated 3-manifolds. There are several examples of pentagon identities arising from supersymmetric gauge theory computations, see, e.g. [6–15].

3 Bailey pairs

Rogers–Ramanujan type identities are being continuously used in the solution of the integrable models, namely to derive the Yang–Baxter and the pentagon identities. In fact, a well-known example of this usage is conducted during the investigations of the hard hexagon model by Baxter. It turns out that Bailey discovered a systematic way to derive these types of identities [1,2,16,17]. As generalized by Andrews [18,19], there exists an iterative scheme to derive infinitely many of these identities if one pair, called a Bailey pair is known. This forms the so-called Bailey chain. The induction step of generating the particular Bailey pairs is referred to as the Bailey lemma for the chain we consider.

A generalization of the Bailey pairs approach to the integral identities is firstly done by Spiridonov in [20,21]. The construction of integral Bailey pairs yields new powerful verifications of various supersymmetric dualities [22,23], generating solutions to the Yang-Baxter equation [24–27], etc.

Accordingly, the generalized version of the Bailey chain is a couple of infinite sequences of holomorphic functions $\{\alpha_n^{(i)}\}_{n\geq 0}$ and $\{\beta_n^{(i)}\}_{n\geq 0}$ such that there exists an identity independent of i which connect $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ as

$$\beta_n^{(i)} = F_n(\alpha_0^{(i)}, \alpha_1^{(i)}, \ldots, \alpha_n^{(i)}), \tag{3.1}$$

where F can be an operator which may now include sum or integrals. Here, $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ are constructed according to

$$\alpha_n^{(i)} = G(\alpha_0^{(i)}, \alpha_1^{(i)}, \ldots, \alpha_{n-1}^{(i)}), \tag{3.2}$$
$$\beta_n^{(i)} = H(\beta_0^{(i)}, \beta_1^{(i)}, \ldots, \beta_{n-1}^{(i)}), \tag{3.3}$$

where G and H represent integral-sum operators.

Definition 3.1 Let $\{\alpha_m(z;t)\}_{m\in\mathbb{Z}}$ and $\{\beta_m(z;t)\}_{m\in\mathbb{Z}}$ be two sequences of functions. They are said to form a Bailey pair with respect to the parameter t iff

$$\beta_m(w;t) = \sum_{n\in\mathbb{Z}} \int dz \mathcal{B}[twz^{-1}, m-n$$
$$+n_t, tw^{-1}z, -m+n+n_t]\, \alpha_n(z;t)\,. \tag{3.4}$$

[3] In this case parameters a_i, b_i, and z stand for the continuous spin variables.

[4] Yet, as $SU(3)\times SU(3)\times U(1)$ has five independent parameters, the above form must be correct even for the more general balancing conditions in (1.4, 1.5).

Lemma 3.1 *If $\{\alpha_m(z;t)\}_{m\in\mathbb{Z}}$ and $\{\beta_m(z;t)\}_{m\in\mathbb{Z}}$ form a Bailey pair with respect to t, then the following sequences*

$$\alpha'_n(w;st) = \mathcal{B}[tuw, n+n_u+n_t, s^2, 2n_s]\alpha_n(w;t) \tag{3.5}$$

$$\begin{aligned}\beta'_n(w;st) = \sum_{m\in\mathbb{Z}} \int \frac{dx}{2\pi i x}\mathcal{B}[swx^{-1}, -m+n \\ + n_s; ux, n_u+m]\mathcal{B}[st^2uw, n+2n_t \\ + n_u+n_s, sw^{-1}x, -n+m+n_s]\beta_m(x;t)\end{aligned} \tag{3.6}$$

form a Bailey pair with respect to the parameter st.

Proof We have to show that

$$\begin{aligned}\beta'_n(w,st) = \sum_{p\in\mathbb{Z}} \int \mathcal{B}[stwy^{-1}, n-p+n_s \\ + n_t, sty^{-1}x, -n+p+n_s+n_t]\alpha'_p(y,st)dy.\end{aligned} \tag{3.7}$$

Inserting (3.4) in (3.6), we first calculate the left-hand side of the equality (3.7)

$$\begin{aligned}\beta'_n(w;st) =& \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \frac{dx}{2\pi i x}\mathcal{B}[swx^{-1}, n+n_s-m, ux, m+n_u] \\ &\times \mathcal{B}[st^2uw, n+n_u+2n_t+n_s, sw^{-1}x, m-n+n_s] \\ &\times \sum_{p\in\mathbb{Z}} \int_{\mathbb{T}} \mathcal{B}[txy^{-1}, m-p \\ &+ n_t, tx^{-1}y, -m+p+n_t]\alpha_p(y,t)dy \\ =& \sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \int\int \mathcal{B}[swx^{-1}, -m+n+n_s, ux, m+n_u] \\ &\times \mathcal{B}[st^2uw, n+n_u+2n_t+n_s, sw^{-1}x, -n+m+n_s] \\ &\times \mathcal{B}[txy^{-1}, m-p+n_t, tx^{-1}y, -m+p+n_t] \\ &\times \alpha_p(y,t)dy\frac{dx}{2\pi i x}\end{aligned} \tag{3.8}$$

Hence, by regrouping the terms accordingly, we obtain[5]

$$\begin{aligned}&\sum_{p\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int (-q^{\frac{1}{2}})^{\frac{|m+n_u|}{2}+\frac{|m-n+n_s|}{2}+\frac{|m-p+n_t|}{2}+\frac{|n-m+n_s|}{2}+\frac{|m-n_u|}{2}+\frac{|p-m+n_t|}{2}} \\ &\times (ux)^{-\frac{|m+n_u|}{2}}(sw^{-1}x)^{-\frac{|m-n+n_s|}{2}}(ty^{-1}x)^{-\frac{|m-p+n_t|}{2}}(swx^{-1})^{-\frac{|n-m+n_s|}{2}}(s^2t^2q^{-1}ux)^{\frac{|m-n_u|}{2}}(tyx^{-1})^{-\frac{|p-m+n_t|}{2}} \\ &\times \frac{(q^{1+\frac{|n-m+n_s|}{2}}(swx^{-1})^{-1})_\infty}{(q^{\frac{|n-m+n_s|}{2}}swx^{-1})_\infty}\frac{(q^{1+\frac{|m+n_u|}{2}}(ux)^{-1})_\infty}{(q^{\frac{|m+n_u|}{2}}ux,)_\infty}\frac{(q^{1+\frac{|m-n+n_s|}{2}}(sw^{-1}x)^{-1})_\infty}{(q^{\frac{|m-n+n_s|}{2}}sw^{-1}x)_\infty}\frac{(q^{1+\frac{|m-n_u|}{2}}s^2t^2q^{-1}ux)_\infty}{(q^{\frac{|m-n_u|}{2}}s^{-2}t^{-2}qu^{-1}x^{-1})_\infty} \\ &\times \frac{(q^{1+\frac{|m-p+n_t|}{2}}(ty^{-1}x)^{-1})_\infty}{(q^{\frac{|m-p+n_t|}{2}}ty^{-1}x)_\infty}\frac{(q^{1+\frac{|p-m+n_t|}{2}}(tyx^{-1})^{-1})_\infty}{(q^{\frac{|p-m+n_t|}{2}}tyx^{-1})_\infty}(-q^{\frac{1}{2}})^{-\frac{|n-n_t|}{2}+|n_t|+\frac{|n+n_t|}{2}}(swu)^{\frac{|n-n_t|}{2}}(st^2uw)^{-\frac{|n+n_t|}{2}}(q^{-1}t^2)^{|n_t|} \\ &\times \frac{(q^{1+\frac{|n-n_t|}{2}}swq^{-1}u)_\infty}{(q^{\frac{|n-n_t|}{2}}(swq^{-1}u)^{-1})_\infty}\frac{(q^{1+\frac{|n+n_t|}{2}}(st^2uw)^{-1})}{(q^{\frac{|n+n_t|}{2}}st^2uw)_\infty}\frac{(q^{1+|n_t|}q^{-1}t^2)_\infty}{(q^{|n_t|}qt^{-2})_\infty}\alpha_p(y,t)dy\frac{dx}{2\pi i x}\end{aligned} \tag{3.9}$$

where we required the sum of the powers of x to vanish, namely

$$n_u+n_s+n_t = 0 \tag{3.10}$$

Upon renaming the variables as

$$a_1 = u \to m_1 = n_u \quad b_1 = sw \to n_1 = n+n_s \tag{3.11}$$

$$a_2 = sw^{-1} \to m_2 = -n+n_s \quad b_2 = qs^{-2}t^{-2}u^{-1} \to n_2 = n_u \tag{3.12}$$

$$a_3 = ty^{-1} \to m_3 = -p+n_t \quad b_3 = tx \to n_3 = p+n_t \tag{3.13}$$

we identify the integral relation (1.3). Also, observe that the constraint (3.10) resulted in the balancing condition (1.5). We hence get upon simplification and regrouping of the terms

$$\begin{aligned}&\sum_{p\in\mathbb{Z}} \int \alpha_p(y,t)dy(-q^{\frac{1}{2}})^{\frac{|n-p-n_u|}{2}+\frac{|-n+p-n_u|}{2}-|n_u|} \\ &\times (stwy^{-1})^{-\frac{|n-p-n_u|}{2}}(stw^{-1}y)^{-\frac{|p-n-n_u|}{2}}(s^2t^2)^{|n_u|} \\ &\times \frac{(q^{1+\frac{|n-p-n_u|}{2}}(stwy^{-1})^{-1})_\infty}{(q^{\frac{|n-p-n_u|}{2}}stwy^{-1})_\infty}\frac{(q^{1+\frac{|p-n-n_u|}{2}}(stw^{-1}y)^{-1})_\infty}{(q^{|p-n-n_u|}stw^{-1}y)_\infty} \\ &\times \frac{(q^{\frac{|n_u|}{2}}s^2t^2)_\infty}{(q^{1+\frac{|n_u|}{2}}s^{-2}t^{-2})_\infty} \\ &\times (-q^{\frac{1}{2}})^{\frac{|p-n_s|}{2}+|n_s|-\frac{|n_s+p|}{2}}(tuy)^{-\frac{|p-n_s|}{2}}(s^2)^{-|n_s|}(s^2tuy)^{\frac{|p+n_s|}{2}} \\ &\times \frac{(q^{1+\frac{|p-n_s|}{2}}(tuy)^{-1})_\infty}{(q^{\frac{|p-n_s|}{2}}tuy)_\infty}\frac{(q^{1+|n_s|}s^{-2})_\infty}{(q^{|n_s|}s^2)_\infty}\frac{(q^{1+\frac{|p+n_s|}{2}}s^2tuy)_\infty}{(q^{\frac{|p+n_s|}{2}}(s^2tuy)^{-1})_\infty},\end{aligned} \tag{3.14}$$

[5] For convenience q of the q-product is omitted.

which is the desired operator equality

$$\sum_{p\in\mathbb{Z}}\int dy\mathcal{B}[stwy^{-1}, n_s+n_t+n-p, stw^{-1}y, n_s +n_t-n+p)\mathcal{B}[tyu, n_t+p+n_u, s^2, 2n_s] = \sum_{p\in\mathbb{Z}}\int dy\mathcal{B}[stwy^{-1}, n_s+n_t +n-p, stw^{-1}y, n_s+n_t-n+p). \tag{3.15}$$

□

4 Conclusions

In this work, we have constructed a new integral Bailey pair for the pentagon identity in the form of q-hypergeometric functions. One can use this Bailey construction to obtain new supersymmetric dualities for linear quiver theories. Namely, any relation between Bailey pairs $\alpha^{(n)}$ and $\beta^{(n)}$ gives integral identities corresponding to the equality of partition functions of certain dual linear quivers, see e.g. [22,23].

We would like to mention that the pentagon identity presented here can also be written as the star-triangle relation for some integrable lattice model of statistical mechanics. It would be interesting to construct the Bailey pairs corresponding to the star-triangle form of the same integral identity.

Acknowledgements The authors are grateful to Mustafa Mullahasanoglu, Sena Ergisi, Dilara Kosva, and Batuhan Keskin for valuable discussions related to the construction of the Bailey pairs and pentagon identities. The work of Ilmar Gahramanov is partially supported by the Bogazici University Research Fund under grant number 20B03SUP3, and TUBITAK grants 220N106 and 122F451.

Data availability This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

Funded by SCOAP[3]. SCOAP[3] supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

1. W.N. Bailey, Some identities in combinatory analysis. Proc. Lond. Math. Soc. **2**(1), 421–435 (1946)
2. W.N. Bailey, Identities of the Rogers–Ramanujan Type. Proc. Lond. Math. Soc. **2**(1), 1–10 (1948)
3. R. Kashaev, F. Luo, G. Vartanov, A TQFT of Turaev–Viro type on shaped triangulations. Ann. Henri Poincare **17**(5), 1109–1143 (2016). arXiv:1210.8393 [math.QA]
4. R.M. Kashaev, Beta pentagon relations. Theor. Math. Phys. **181**(1), 1194–1205 (2014). arXiv:1403.1298 [math-ph]
5. R. Kashaev, Euler's beta function and pentagon relations. Acta Math. Vietnam **39**(4), 561–566 (2014)
6. I. Gahramanov, H. Rosengren, A new pentagon identity for the tetrahedron index. JHEP **11**, 128 (2013). arXiv:1309.2195 [hep-th]
7. I. Gahramanov, H. Rosengren, Integral pentagon relations for 3d superconformal indices. arXiv:1412.2926 [hep-th]. [Proc. Symp. Pure Math. **93**, 165 (2016)]
8. I. Gahramanov, H. Rosengren, Basic hypergeometry of supersymmetric dualities. Nucl. Phys. B **913**, 747–768 (2016). arXiv:1606.08185 [hep-th]
9. D.N. Bozkurt, I. Gahramanov, Pentagon identities arising in supersymmetric gauge theory computations. Teor. Mat. Fiz. **198**(2), 215–224 (2019). arXiv:1803.00855 [math-ph]
10. D.N. Bozkurt, I. Gahramanov, M. Mullahasanoglu, Lens partition function, pentagon identity, and star-triangle relation. Phys. Rev. D **103**(12), 126013 (2021). arXiv:2009.14198 [hep-th]
11. S. Jafarzade, New pentagon identities revisited. J. Phys. Conf. Ser. **1194**(1), 012054 (2019). arXiv:1812.01325 [math-ph]
12. T. Dimofte, D. Gaiotto, S. Gukov, 3-manifolds and 3d indices. Adv. Theor. Math. Phys. **17**(5), 975–1076 (2013). arXiv:1112.5179 [hep-th]
13. T. Dimofte, D. Gaiotto, S. Gukov, Gauge theories labelled by three-manifolds. Commun. Math. Phys. **325**, 367–419 (2014). arXiv:1108.4389 [hep-th]
14. D. Gang, E. Koh, S. Lee, J. Park, Superconformal index and 3d–3d correspondence for mapping cylinder/torus. JHEP **01**, 063 (2014). arXiv:1305.0937 [hep-th]
15. S. Benvenuti, S. Pasquetti, 3d $\mathcal{N}=2$ mirror symmetry, pq-webs and monopole superpotentials. JHEP **08**, 136 (2016). arXiv:1605.02675 [hep-th]
16. S.O. Warnaar, 50 years of bailey's lemma. In: Algebraic combinatorics and applications, pp. 333–347 (2001). arXiv:0910.2062 [math.CO]
17. W. Zudilin, Hypergeometric heritage of W. N. Bailey. Not. Int. Congr. Chin. Math. **7**(2), 32–46 (2019)
18. G. Andrews, Multiple series Rogers–Ramanujan type identities. Pac. J. Math. **114**(2), 267–283 (1984)
19. G.E. Andrews, Bailey's transform, lemma, chains and tree. In: Special Functions 2000: Current Perspective and Future Directions, pp. 1–22 (2001)
20. V. Spiridonov, An elliptic incarnation of the bailey chain. Int. Math. Res. Not. **2002**(37), 1945–1977 (2002)
21. V.P. Spiridonov, A bailey tree for integrals. Theor. Math. Phys. **139**(1), 536–541 (2004). arXiv:math/0312502
22. F. Brünner, V.P. Spiridonov, A duality web of linear quivers. Phys. Lett. B **761**, 261–264 (2016). arXiv:1605.06991 [hep-th]
23. F. Brünner, V.P. Spiridonov, 4d $\mathcal{N}=1$ quiver gauge theories and the A_n Bailey lemma. JHEP **03**, 105 (2018). arXiv:1712.07018 [hep-th]
24. S.E. Derkachov, V.P. Spiridonov, Yang–Baxter equation, parameter permutations, and the elliptic beta integral. Russ. Math. Surv. **68**, 1027–1072 (2013). arXiv:1205.3520 [math-ph]
25. I. Gahramanov, V.P. Spiridonov, The star-triangle relation and 3d superconformal indices. JHEP **08**, 040 (2015). arXiv:1505.00765 [hep-th]
26. K.Y. Magadov, V.P. Spiridonov, Matrix Bailey lemma and the star-triangle relation. SIGMA **14**, 121 (2018)
27. V.P. Spiridonov, The rarefied elliptic Bailey lemma and the Yang–Baxter equation. J. Phys. A **52**(35), 355201 (2019). arXiv:1904.12046 [math-ph]