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Self-adjoint and non-self-adjoint extensions of symmetric q-Sturm–Liouville operators

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Abstract. A space of boundary values is constructed for minimal symmetric regular and singular q-Sturm–Liouville operators in limit-point and limit-circle cases. A description of all maximal dissipative, maximal accumulative, self-adjoint, and other extensions of such symmetric q-Sturm–Liouville operators is given in terms of boundary conditions.

1. Introduction and notations

In this section, we introduce some of the needed q-notations and results (see [2-4, 6-9, 10, 11]). Throughout this paper, q is a positive number with $0 < q < 1$. For $\mu \in \mathbb{R} := (-\infty, \infty)$, a set $A \subseteq \mathbb{R}$ is called a μ-geometric set if $\mu t \in A$ for all $t \in A$. If $A \subseteq \mathbb{R}$ is a μ-geometric set, then it contains all geometric sequences $\{\mu^n t\}$ $(n = 0, 1, 2...)$, $t \in A$. Let f be a function, real or complex-valued, defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(t) := \frac{f(t) - f(qt)}{t - qt}, \ t \in A \backslash \{0\}. \tag{1.1}$$

If $0 \in A$, the q-derivative at zero is defined by

$$D_q f(0) := \lim_{n\to\infty} \frac{f(q^n t) - f(0)}{q^n t},$$

if the limit exists and does not depend on t. Since the formulation of the extension problems requires the definition of $D_{q^{-1}}$ in a same manner to be

$$D_{q^{-1}} f(t) := \begin{cases} \frac{f(t)-f(q^{-1}t)}{t-q^{-1}t}, \ t \in A\backslash\{0\}, \\ D_q f(0), \ t = 0, \end{cases}$$

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provided that $D_q f(0)$ exists. As a converse of the q-difference operator, Jackson's q-integration [14], is given by

$$\int_0^x f(t)d_qt := x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x),\ x \in A,$$

provided that the series converges, and

$$\int_a^b f(t)d_qt := \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,\ a, b \in A.$$

The q-integration for a function over $[0,\infty)$ defined by the formula ([5, 7])

$$\int_0^{\infty} f(t)\, d_qt = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

When required q will be replaced by q^{-1}. The following facts can be verified directly from the definition and will be used often:

$$D_{q^{-1}} f(t) = (D_q)f(q^{-1}t),\ D_q^2 f(q^{-1}t) = qD_q[D_q f(q^{-1}t)] = D_{q^{-1}} D_q f(t).$$

Associated with this operator, there is a non-symmetric formula for the q-differentation of a product

$$D_q[f(t)g(t)] = f(qt)D_q g(t) + g(t)D_q f(t).$$

We will deal only with functions q-regular at zero, that is, functions satisfying

$$\lim_{n\to\infty} f(q^n t) = f(0).$$

The class of the functions which are q-regular at zero includes continuous functions. If f and g are q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^a g(t)D_q f(t)d_qt = (fg)(a) - (fg)(0) - \int_0^a D_q g(t) f(qt)d_qt.$$

The q-difference calculus or quantum calculus was initiated in the beginning of the 19th century. Since then, the subject of q-differential equations has evolved into a multidisciplinary subject ([11, 12, 15]). There are several physical models involving q-derivatives, q-integrals, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and q-Gamma functions, q-Euler–Maclaurin formula and their related problems (see [6, 11, 12, 14]).

In [9], Annaby and Mansour studied a q-Sturm–Liouville eigenvalue problem and formulated a self-adjoint q-Sturm–Liouville operator in a Hilbert space. They also discussed properties of the eigenvalues and the eigenfunctions. Annaby et al. [7] established the q-Titchmarsh–Weyl theory for singular q-Sturm–Liouville problems and defined q-limit-point and q-limit-circle singularities. In [4, 8], the authors gave sufficient conditions which assure that the singular point is in a limit-point case.

The theory of extensions of symmetric operators is one of the main branches in operator theory which is closely related to various fields of mathematics. In operator theory, there exists an abstract scheme of constructing maximal dissipative (self-adjoint) extensions of symmetric operators that are parametrized by contraction (unitary) operators (see [1, 10, 13, 16-18]). However, regardless of the general scheme, the problem of the description of the maximal dissipative (accumulative), self-adjoint and other extensions of a given symmetric operator via the boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular operators, because at the singular ends of the interval under consideration the usual boundary conditions are meaningless in general ([1, 3, 5, 10, 13, 16]).

In this paper, we consider the minimal symmetric regular and singular q-Sturm–Liouville operators in Weyl's limit-point and limit-circle cases. We construct a space of boundary values and describe all maximal dissipative, maximal accumulative, self-adjoint and other extensions of symmetric q-Sturm–Liouville operators in terms of the boundary conditions.

2. Self-adjoint and non-self-adjoint extensions of the symmetric regular q-Sturm–Liouville operator

We consider the regular q-Sturm–Liouville expression

$$(Lx)(t) = -\frac{1}{q} D_{q^{-1}} D_q x(t) + u(t)x(t), \tag{2.1}$$

where $t \in \mathbb{I} := [0, a]$, $0 < a < \infty$, u is a real-valued function defined on $[0, q^{-1}a]$ and q-regular at zero and D_q is the q-difference operator defined in (1.1).

To pass from the expression (2.1) to operators, we introduce the Hilbert space $\mathcal{L}_q^2(\mathbb{I})$ consisting of all complex-valued functions x such that

$$\int_0^a |x(t)|^2 \, d_q t < \infty$$

with the inner product

$$(x, y) = \int_0^a x(t)\overline{y(t)} d_q t.$$

Denote by $\mathcal{D}_{\max}$ the linear set of all functions $x \in \mathcal{L}_q^2(\mathbb{I})$ such that x and $D_q x$ are q-regular functions at 0 and $Lx \in \mathcal{L}_q^2(\mathbb{I})$. We define the *maximal* operator $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max} x = Lx$. For each $x, y \in \mathcal{D}_{\max}$ we define the *q-Wronski determinant* (or *q-Wronskian*)

$$\mathcal{W}_q[x, y](t) = x(t) D_{q^{-1}} y(t) - D_{q^{-1}} x(t) y(t), t \in \mathbb{I}.$$

For two arbitrary functions $x, y \in \mathcal{D}_{\max}$, we have q-Green's formula (or Lagrange's identity) ([2, 3, 6])

$$\int_0^t (Lx)(\xi))\overline{y(\xi)} d_q \xi - \int_0^t x(\xi)(\overline{Ly)(\xi)} d_q \xi = [x, y](t) - [x, y](0), \ t \in \mathbb{I}, \tag{2.2}$$

where

$$[x, y](t) := \mathcal{W}_q[x, \overline{y}](t) = x(t)\overline{D_{q^{-1}} y(t)} - D_{q^{-1}} x(t)\overline{y(t)}, \ t \in \mathbb{I}.$$

For any function $x \in \mathcal{D}_{\max}$, $x(0)$ and $D_{q^{-1}} x(0)$ can be defined by $x(0) := \lim_{t \to 0^+} x(t)$ and $D_{q^{-1}} x(0) := \lim_{t \to 0^+} D_{q^{-1}} x(t)$. These limits exist and are finite (since x and $D_{q^{-1}} x$ are q-regular function at 0).

It is known that (for detail see [1, 10, 16]) a linear symmetric operator A acting from a Hilbert space H into H has always a pair integers (n^+, n^-) which are non-negative integers or $+\infty$. This pair is associated with the dimensions of the spaces $\mathcal{N}^+$ and $\mathcal{N}^-$, respectively, where

$$\mathcal{N}_\lambda = \{y \in D(A^*) : A^* y = \lambda y\}, \ \lambda \in \mathbb{C},$$

$\mathcal{N}^+ = \mathcal{N}_i$, $\mathcal{N}^- = \mathcal{N}_{-i}$, $n^+ = \dim \mathcal{N}^+$, $n^- = \dim \mathcal{N}^-$ and the pair (n^+, n^-) is called the deficiency indices of A.

The formula

$$D(A^*) = D(A) + \mathcal{N}^+ + \mathcal{N}^-,$$

holds (see [1, 10, 16]).

In $\mathcal{L}_q^2(\mathbb{I})$, we consider the linear dense set $\mathcal{D}_{\min}$ consisting of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$x(0) = D_{q^{-1}} x(0) = 0, \ (x)(a) = D_{q^{-1}} x(a) = 0.$$

Denote by $\mathcal{L}_{\min}$ the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$. It follows from (2.2) that $\mathcal{L}_{\min}$ is symmetric. The *minimal operator* $\mathcal{L}_{\min}$ is a closed, symmetric operator with deficiency indices $(2, 2)$, and $\mathcal{L}_{\max} = \mathcal{L}_{\min}^*$

Recall that a linear operator S (with dense domain $\mathcal{D}(S)$) acting in some Hilbert space H is called *dissipative (accumulative)* if $\Im(Sf, f) \geq 0$ ($\Im(Sf, f) \leq 0$) for all $f \in \mathcal{D}(S)$ and *maximal dissipative (maximal accumulative)* if it does not have a proper dissipative (accumulative) extension ([13]).

An important role in the theory of extensions is played by the concept of the space of boundary values of the symmetric operator. The triplet $(\mathcal{H}, \Gamma_1, \Gamma_2)$, where $\mathcal{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear mappings of $\mathcal{D}(A^*)$ into $\mathcal{H}$, is called (see [13, p.152] a *space of boundary values* of a closed symmetric operator A acting in a Hilbert space H with equal (finite or infinite) deficiency indices if

(i) $(A^*f, g)_H - (f, A^*g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathcal{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathcal{H}}, \forall f, g \in \mathcal{D}(A^*)$,

and

(ii) for every $F_1, F_2 \in \mathcal{H}$, there exists a vector $f \in \mathcal{D}(A^*)$ such that $\Gamma_1 f = F_1$ and $\Gamma_2 f = F_2$.

We denote by Θ_1 and Θ_2 the linear mappings of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$ defined by

$$\Theta_1 x = \begin{pmatrix} -x(0) \\ x(a) \end{pmatrix}, \Theta_2 x = \begin{pmatrix} D_{q^{-1}}x(0) \\ D_{q^{-1}}x(a) \end{pmatrix}. \tag{2.3}$$

Then we have the next result.

Theorem 2.1. *The triplet $\left(\mathbb{C}^2, \Theta_1, \Theta_2\right)$ defined according to (2.3) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Proof. The first requirement of the definition of a space of boundary values holds in view $(\mathcal{L}_{\max}x, y) - (x, \mathcal{L}_{\max}y) = [x, y](a) - [x, y](0)$ $(\forall x, y \in \mathcal{D}_{\max})$ and

$$(\Theta_1 x, \Theta_2 y)_{\mathbb{C}^2} - (\Theta_2 x, \Theta_1 y)_{\mathbb{C}^2} = -x(0)\overline{D_{q^{-1}}y(0)} + x(a)\overline{D_{q^{-1}}y(a)}$$

$$+D_{q^{-1}}x(0)\overline{y(0)} - D_{q^{-1}}x(a)\overline{y(a)} = (\mathcal{L}_{\max}x, y) - (x, \mathcal{L}_{\max}y), \ \forall x, y \in \mathcal{D}_{\max}.$$

The second requirement will be proved as to following lemma.

Lemma 2.2. *For any complex numbers α, β, γ and δ there is a function $x \in \mathcal{D}_{\max}$ satisfying the conditions*

$$x(0) = \alpha, D_{q^{-1}}x(0) = \beta, x(a) = \gamma, D_{q^{-1}}x(a) = \delta. \tag{2.4}$$

Proof. We denote by $\varphi(t)$ and $\psi(t)$ the solutions (real-valued) of the equation

$$Lx = 0, \ t \in \mathbb{I} \tag{2.5}$$

satisfying the initial conditions

$$\varphi(0) = 1, D_{q^{-1}}\varphi(0) = 0, \psi(0) = 0, D_{q^{-1}}\psi(0) = 1. \tag{2.6}$$

The q-Wronskian of the two solutions of (2.5) does not depend on t, and the two solutions of this equation are linearly independent if and only if their q-Wronskian is non-zero. It follows from the conditions (2.6) and the constancy of the q-Wronskian that ([2, 3, 6, 9])

$$\mathcal{W}_q[\varphi, \psi](t) = \mathcal{W}_q[\varphi, \psi](0) = 1 \ (t \in \mathbb{I}). \tag{2.7}$$

Consequently, φ and ψ form a fundamental system of solutions of (2.5).

Let y be an arbitrary vector in $\mathcal{L}_q^2(\mathbb{I})$ satisfying

$$(y, \varphi) = \beta, (y, \psi) = -\alpha. \tag{2.8}$$

There is such a y even among the linear combination of φ and ψ. Indeed, if we set $y = c_1\varphi + c_2\psi$, then conditions (2.8) are a system of equations for constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent functions φ and ψ and is therefore non-zero.

Denote by $x_1(t)$ the solution of $L(x) = y(t)$ $(t \in [0, a])$ satisfying the initial conditions $x_1(a) = 0, D_{q^{-1}}x_1(a) = 0$. Then we have $x_1 \in \mathcal{D}_{\max}$. Further, applying Green's formula (2.2) to x_1 and φ, we obtain

$$(y, \varphi) = (L(x_1), \varphi) = [x_1, \varphi](a) - [x_1, \varphi](0) + (x_1, L(\varphi)).$$

But $L(\varphi) = 0$, and thus $(x_1, L(\varphi)) = 0$. Moreover, since $x_1(a) = 0, D_{q^{-1}}x_1(a) = 0$, we have

$$[x_1, \varphi](0) = x_1(0) D_{q^{-1}}\varphi(0) - D_{q^{-1}}x_1(0)\varphi(0) = -D_{q^{-1}}x_1(0) = -\beta.$$

Analogously,

$$-(y, \psi) = -(L(x_1), \psi) = -[x_1, \psi](a) + [x_1, \psi](0) - (x_1, L(\psi))$$

$$= x_1(0) D_{q^{-1}}\psi(0) - D_{q^{-1}}x_1(0)\psi(0) = x_1(0) = \alpha.$$

Therefore

$$x_1(0) = \alpha, D_{q^{-1}}x_1(0) = \beta, x_1(a) = 0, D_{q^{-1}}x_1(a) = 0. \tag{2.9}$$

Similarly, denote by x_2 ($x_2 \in \mathcal{D}_{\max}$) the function satisfying the conditions

$$x_2(0) = 0, D_{q^{-1}}x_2(0) = 0, x_2(a) = \gamma, D_{q^{-1}}x_2(a) = \delta.$$

Now let us define the function $x = x_1 + x_2$. It is clear that $x \in \mathcal{D}_{\max}$, and

$$x(0) = \alpha, D_{q^{-1}}x(0) = \beta, x(a) = \gamma, D_{q^{-1}}x(a) = \delta,$$

then Lemma 2.2 is proved. □

Using Theorem 2.1 and [13] we can formulate the following theorem.

Theorem 2.3. *For any contraction S in $\mathbb{C}^2$ the restriction of the operator $\mathcal{L}_{\max}$ to the set of functions $x \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$(S - I)\Theta_1 x + i(S + I)\Theta_2 x = 0 \tag{2.10}$$

or

$$(S - I)\Theta_1 x - i(S + I)\Theta_2 x = 0 \tag{2.11}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $\mathcal{L}_{\min}$. Conversely, every maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$ is the restriction of $\mathcal{L}_{\max}$ to the set of vectors $x \in \mathcal{D}_{\max}$ satisfying (2.10) ((2.11)), and the contraction S is uniquely determined by the extensions. These conditions define a self-adjoint extension if and only if S is unitary. In the latter case (2.10) and (2.11) are, equivalent to the condition $(\cos A)\Theta_1 x - (\sin A)\Theta_2 x = 0$, where A is a Hermitian matrix in $\mathbb{C}^2$. The general form of the dissipative and accumulative extensions of the operator $\mathcal{L}_{\min}$ is given by the conditions

$$S(\Theta_1 x + i\Theta_2 x) = \Theta_1 x - i\Theta_2 x,\ \Theta_1 x + i\Theta_2 x \in \mathcal{D}(S) \tag{2.12}$$

$$S(\Theta_1 x - i\Theta_2 x) = \Theta_1 x + i\Theta_2 x,\ \Theta_1 x - i\Theta_2 x \in \mathcal{D}(S) \tag{2.13}$$

respectively, where S is a linear operator in $\mathbb{C}^2$ with $\|Sx\| \le \|x\|$, $x \in \mathcal{D}(S)$. The general form of symmetric extensions is given by the formulas (2.12) and (2.13), where S is an isometric operator.

In particular, the boundary conditions ($x \in \mathcal{D}_{\max}$)

$$D_{q^{-1}}x(0) - \omega_1 x(0) = 0, \tag{2.14}$$

$$x(a) - \omega_2 D_{q^{-1}}x(a) = 0 \tag{2.15}$$

with $\Im\omega_1 \ge 0$ or $\omega_1 = \infty$, and $\Im\omega_2 \ge 0$ or $\omega_2 = \infty$ ($\Im\omega_1 \le 0$ or $\omega_1 = \infty$, and $\Im\omega_2 \le 0$ or $\omega_2 = \infty$) describe all maximal dissipative (maximal accumulative) extensions of $\mathcal{L}_{\min}$ with separated boundary conditions. The self-adjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when $\Im\omega_1 = 0$ or $\omega_1 = \infty$, and $\Im\omega_2 = 0$ or $\omega_2 = \infty$. Here for $\omega_1 = \infty$ ($\omega_2 = \infty$), condition (2.14) ((2.15)) should be replaced by $x(0) = 0$ ($D_{q^{-1}}x(a) = 0$).

3. Self-adjoint and non-self-adjoint extensions of the symmetric singular q-Sturm–Liouville operators

We consider the singular q-Sturm–Liouville expression

$$(Lx)(t) = -\frac{1}{q} D_{q^{-1}} D_q x(t) + u(t)x(t),\ t \in \mathcal{I} := [0, \infty), \tag{3.1}$$

where u is a real-valued function defined on $\mathcal{I}$ and q-regular at zero and D_q is the q-difference operator defined in (1.1).

To pass from the expression (3.1) to operators, we introduce the Hilbert space $\mathcal{L}_q^2(\mathcal{I})$ consisting of all complex-valued functions x such that

$$\int_0^\infty |x(t)|^2 \, d_q t < \infty$$

with the inner product

$$(x, y) = \int_0^\infty x(t)\overline{y(t)} d_q t.$$

Denote by $\mathcal{D}_{\max}$ the linear set of all functions $x \in \mathcal{L}_q^2(\mathcal{I})$ such that x and $D_q x$ are q-regular functions at 0 and $Lx \in \mathcal{L}_q^2(\mathcal{I})$. We define the *maximal operator* $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max} x = Lx$. It is clear from (2.2) that limit $[x, y](\infty) := \lim_{t\to\infty} [x, y](t)$ exists and is finite for all $x, y \in \mathcal{D}_{\max}$. For any function $x \in \mathcal{D}_{\max}$, $x(0)$ and $D_{q^{-1}} x(0)$ can be defined by $x(0) := \lim_{t\to 0^+} x(t)$ and $D_{q^{-1}} x(0) := \lim_{t\to 0^+} D_{q^{-1}} x(t)$. These limits exist and are finite (since x and $D_{q^{-1}} x$ are q-regular at zero).

In $\mathcal{L}_q^2(\mathcal{I})$, we consider the linear dense set $\mathcal{D}_{\min}$ consisting of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$x(0) = D_{q^{-1}} x(0) = 0,\ [x, y](\infty) = 0,\ \forall y \in \mathcal{D}_{\max}. \tag{3.2}$$

Denote by $\mathcal{L}_{\min}$ the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$. It follows from (3.2) that $\mathcal{L}_{\min}$ is symmetric. The *minimal operator* $\mathcal{L}_{\min}$ is a closed, symmetric operator with deficiency indices (2, 2), and $\mathcal{L}_{\max} = \mathcal{L}_{\min}^*$

(a) Let symmetric operator $\mathcal{L}_{\min}$ has deficiency indices (1, 1), so the case of Weyl's limit-point occurs for $\mathcal{L}_{\min}$ (see [1-4, 6-8, 10, 16]). Then $[x, y](\infty) = 0$ for all $x, y \in \mathcal{D}_{\min}$. The domain $\mathcal{D}_{\min}$ of the symmetric operator $\mathcal{L}_{\min}$ consists of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions $x(0) = D_{q^{-1}} x(0) = 0$.

We denote by Φ_1 and Φ_2 the linear mappings of $\mathcal{D}_{\max}$ into $\mathbb{C}$ defined by

$$\Phi_1 x = -x(0),\ \Phi_2 x = D_{q^{-1}} x(0). \tag{3.3}$$

Then we have the following theorem.

Theorem 3.1. *The triplet* $(\mathbb{C}, \Phi_1, \Phi_2)$ *defined according to (3.3) is a space of boundary values of the operator* $\mathcal{L}_{\min}$.

Proof. The first requirement of the definition of a space of boundary values holds in view

$$(\mathcal{L}_{\max} x, y) - (x, \mathcal{L}_{\max} y) = -[x, y](0)\ \ (\forall x, y \in \mathcal{D}_{\max})$$

and

$$(\Phi_1 x, \Phi_2 y)_{\mathbb{C}} - (\Phi_2 x, \Phi_1 y)_{\mathbb{C}} = -x(0)\overline{D_{q^{-1}} y(0)} + D_{q^{-1}} x(0)\overline{y(0)}$$

$$= -[x, y](0) = (\mathcal{L}_{\max} x, y) - (x, \mathcal{L}_{\max} y),\ \forall x, y \in \mathcal{D}_{\max}.$$

The second requirement will be proved as follows.

Lemma 3.2. *For any complex numbers α and β, there is a function $x \in \mathcal{D}_{\max}$ satisfying the conditions*

$$x(0) = \alpha, D_{q^{-1}} x(0) = \beta.$$

Proof. Let us denote by $\mathcal{L}_{\min,c}$ ($\mathcal{L}_{\max,c}$) the minimal symmetric (maximal) operator generated by L on the set $[0,c]$ $(c \in \mathcal{I})$. Denote by $x_0 \in \mathcal{D}_{\max,c}$ satisfying the conditions (see Lemma 2.2)

$$x(0) = \alpha,\ D_{q^{-1}}x(0) = \beta, x_0(c) = 0,\ D_{q^{-1}}x_0(c) = 0.$$

Then we have $x_0 \in \mathcal{D}_{\max,c}$. Now let us define the function

$$x(t) = \begin{cases} x_0(t),\ 0 \le t \le c, \\ 0,\ c \le t < \infty. \end{cases}$$

It is clear that $x \in \mathcal{D}_{\max}$, and $x(0) = \alpha$, $D_{q^{-1}}x(0) = \beta$, then Lemma 3.2 is proved. □

Using Theorem 3.1 and [13] we can have the following theorem.

Theorem 3.3. *Every maximal dissipative (accumulative) extension $\mathcal{L}_\theta$ of $\mathcal{L}_{\min}$ is determined by the equality $\mathcal{L}_\theta x = \mathcal{L}_{\max}x$ on the vectors $x \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$D_{q^{-1}}x(0) - \theta x(0) = 0, \tag{3.4}$$

where $\Im\theta \ge 0$ or $\theta = \infty$ ($\Im\theta \le 0$ or $\theta = \infty$)*. Conversely, for an arbitrary number θ with $\Im\theta \ge 0$ or $\theta = \infty$* ($\Im\theta \le 0$ or $\theta = \infty$)*, condition (3.4) determines a maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$. The self-adjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when $\Im\theta = 0$* or *$\theta = \infty$. For $\theta = \infty$, the corresponding boundary condition has the form $x(0) = 0$.*

(b) We assume that $\mathcal{L}_{\min}$ have deficiency indices $(2,2)$, so that the limit-circle case holds for the expression L or the operator $\mathcal{L}_{\min}$ (see [1-4, 6-8, 10, 16]). Since $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$, $\varphi, \psi \in \mathcal{L}_q^2(\mathcal{I})$ and, moreover, $\varphi, \psi \in \mathcal{D}_{\max}$.

Lemma 3.4. *For arbitrary functions $x, y \in \mathcal{D}_{\max}$, we have the equality* **(***the Plücker identity***)**

$$[x,y](t) = [x,\varphi](t)[\overline{y},\psi](t) - [x,\psi](t)[\overline{y},\varphi](t),\ t \in \mathcal{I}\cup\{\infty\}. \tag{3.5}$$

Proof. Since the functions φ and ψ are real-valued and since $[\varphi,\psi](t) = 1$ $(t \in \mathcal{I}\cup\{\infty\})$, one obtains

$$\begin{aligned}
&[x,\varphi](t)[\overline{y},\psi](t) - [x,\psi](t)[\overline{y},\varphi](t) \\
&= (xD_{q^{-1}}\varphi - D_{q^{-1}}x\varphi)(t)(\overline{y}D_{q^{-1}}\psi - \overline{D_{q^{-1}}y}\psi)(t) \\
&-(xD_{q^{-1}}\psi - D_{q^{-1}}x\psi)(t)(\overline{y}D_{q^{-1}}\varphi - \overline{D_{q^{-1}}y}\varphi)(t) \\
&= (xD_{q^{-1}}\varphi\overline{y}D_{q^{-1}}\psi - xD_{q^{-1}}\varphi\overline{D_{q^{-1}}y}\psi - D_{q^{-1}}x\varphi\overline{y}D_{q^{-1}}\psi \\
&+D_{q^{-1}}x\varphi\overline{D_{q^{-1}}y}\psi - xD_{q^{-1}}\psi\overline{y}D_{q^{-1}}\varphi + xD_{q^{-1}}\psi\overline{D_{q^{-1}}y}\varphi \\
&+D_{q^{-1}}x\psi\overline{y}D_{q^{-1}}\varphi - D_{q^{-1}}x\psi\overline{D_{q^{-1}}y}\varphi)(t) \\
&= (-x\overline{D_{q^{-1}}y} + D_{q^{-1}}x\overline{y})(t)(D_{q^{-1}}\varphi\psi - \varphi D_{q^{-1}}\psi)(t) = [x,y](t).
\end{aligned}$$

The Lemma 3.4 is proved. □

Theorem 3.5. *The domain $D_{\min}$ of the operator $L_{\min}$ consists of functions $x \in D_{\max}$ satisfying the following boundary conditions*

$$x(0) = D_{q^{-1}}x(0) = 0 = [x,\varphi](\infty) = [x,\psi](\infty) = 0. \tag{3.6}$$

Proof. As noted above, the domain $\mathcal{D}_{\min}$ of $\mathcal{L}_{\min}$ coincides with the set of all functions $x \in \mathcal{D}_{\max}$, satisfying (3.2). By virtue of Lemma 3.4, the conditions (3.2) are equivalent to

$$x(0) = D_{q^{-1}}x(0) = 0, [x,\varphi](\infty)[\overline{y},\psi](\infty) - [x,\psi](\infty)[\overline{y},\varphi](\infty) = 0. \tag{3.7}$$

Further $[\overline{y},\psi](\infty)$ and $[\overline{y},\varphi](\infty)$ $(x \in \mathcal{D}_{\max})$ can be arbitrary, therefore equality (3.7) for all $x \in \mathcal{D}_{\max}$ is possible if and only if the conditions (3.6) hold. The theorem is proved. □

We denote by Θ_1 and Θ_2 the linear mappings of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$ defined by

$$\Theta_1 x = \begin{pmatrix} -x(0) \\ [x,\varphi](\infty) \end{pmatrix}, \ \Theta_2 x = \begin{pmatrix} D_{q^{-1}}x(0) \\ [x,\psi](\infty) \end{pmatrix}. \tag{3.8}$$

Then we have the following theorem.

Theorem 3.6. *The triplet $\left(\mathbb{C}^2, \Theta_1, \Theta_2\right)$ defined according to (3.8) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Proof. The first condition of the definition of a space of boundary values holds in view of (2.2) and Lemma 3.4:

$$(\Theta_1 x, \Theta_2 y)_{\mathbb{C}^2} - (\Theta_2 x, \Theta_1 y)_{\mathbb{C}^2} = -x(0)\, D_{q^{-1}}\overline{y}(0)$$

$$+D_{q^{-1}}x(0)\,\overline{y}(0) + [x,\varphi](\infty)\,[\overline{y},\psi](\infty) - [x,\psi](\infty)\,[\overline{y},\varphi](\infty)$$

$$= [x,y](\infty) - [x,y](0) = (\mathcal{L}_{\max}x, y) - (x, \mathcal{L}_{\max}y), \forall x, y \in \mathcal{D}_{\max}.$$

The second condition will be proved as the following lemma.

Lemma 3.7. *For any complex numbers α, β, γ and δ, there is a function $x \in \mathcal{D}_{\max}$ satisfying the conditions*

$$x(0) = \alpha, \ D_{q^{-1}}x(0) = \beta,$$

$$[x,\varphi](\infty) = \gamma, \ [x,\psi](\infty) = \delta. \tag{3.9}$$

Proof. Let z be an arbitrary vector in $\mathcal{L}_q^2(\mathcal{I})$ satisfying

$$(z,\varphi) = \gamma + \beta, \ (z,\psi) = \delta - \alpha. \tag{3.10}$$

There is such a z even among the linear combination of φ and ψ. Indeed, if we set $z = c_1\varphi + c_2\psi$, then conditions (3.10) form a system of equations for constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent functions φ and ψ and is therefore non-zero.

Denote by $x(t)$ the solution of $L(x) = z(t)$ $(t \in \mathcal{I})$ satisfying the initial conditions $x(0) = \alpha$, $D_{q^{-1}}x(0) = \beta$. We claim that x is the desired function. Applying Green's formula (2.2) to x and φ, we obtain

$$(z,\varphi) = (L(x),\varphi) = [x,\varphi](\infty) - [x,\varphi](0) + (x, L(\varphi)).$$

But $L(\varphi) = 0$, and thus $(x, L(\varphi)) = 0$. Moreover, since $x(0) = \alpha$, $D_{q^{-1}}x(0) = \beta$, we have

$$[x,\varphi](0) = x(0)\, D_{q^{-1}}\varphi(0) - D_{q^{-1}}x(0)\,\varphi(0) = -\beta.$$

Therefore,

$$(z,\varphi) = [x,\varphi](\infty) + \beta. \tag{3.11}$$

Then, from (3.10) and (3.11), we obtain $[x,\varphi](\infty) = \gamma$.

Analogously,

$$(z,\psi) = (L(x),\psi) = [x,\psi](\infty) - [x,\psi](0) + (x, \tau(\psi)) = [x,\psi](\infty) - \alpha. \tag{3.12}$$

Then, from (3.10) and (3.12), we obtain $[x,\psi](\infty) = \delta$. The Lemma 3.7 is proved and consequently, so is Theorem 3.6. □

Using Theorem 3.6 and [13], we can state the following theorem.

Theorem 3.8. *For any contraction T in $\mathbb{C}^2$ the restriction of the operator $\mathcal{L}_{\max}$ to the set of functions $x \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$(T - I)\,\Theta_1 x + i\,(T + I)\,\Theta_2 x = 0 \tag{3.13}$$

or

$$(T-I)\,\Theta_1 x - i\,(T+I)\,\Theta_2 x = 0 \tag{3.14}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $\mathcal{L}_{\min}$. *Conversely, every maximal dissipative (accumulative) extension of* $\mathcal{L}_{\min}$ *is the restriction of* $\mathcal{L}_{\max}$ *to the set of vectors* $x \in \mathcal{D}_{\max}$ *satisfying (3.13) ((3.14)), and the contraction T is uniquely determined by the extensions. These conditions define a self-adjoint extension if and only if T is unitary. In the latter case (3.13) and (3.14) are equivalent to the condition* $(\cos A)\,\Theta_1 x - (\sin A)\,\Theta_2 x = 0$, *where A is a Hermitian matrix in* $\mathbb{C}^2$. *The general form of the dissipative and accumulative extensions of the operator* $\mathcal{L}_{\min}$ *is given by the conditions*

$$T\,(\Theta_1 x + i\Theta_2 x) = \Theta_1 x - i\Theta_2 x, \Theta_1 x + i\Theta_2 x \in \mathcal{D}\,(T) \tag{3.15}$$

$$T\,(\Theta_1 x - i\Theta_2 x) = \Theta_1 x + i\Theta_2 x, \Theta_1 x - i\Theta_2 x \in \mathcal{D}\,(T) \tag{3.16}$$

respectively, where T is a linear operator in $\mathbb{C}^2$ *with* $\|Tx\| \le \|x\|\,, x \in \mathcal{D}\,(T)$. *The general form of symmetric extensions is given by the formulas (3.15) and (3.16), where T is an isometric operator.*

In particular, the boundary conditions $(x \in \mathcal{D}_{\max})$

$$D_{q^{-1}} x\,(0) - \alpha_1 x\,(0) = 0, \tag{3.17}$$

$$[x,\varphi]\,(\infty) - \alpha_2\,[x,\psi]\,(\infty) = 0 \tag{3.18}$$

with $\Im\alpha_1 \ge 0$ *or* $\alpha_1 = \infty$, *and* $\Im\alpha_2 \ge 0$ *or* $\alpha_2 = \infty$ ($\Im\alpha_1 \le 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 \le 0$ or $\alpha_2 = \infty$) *describe all maximal dissipative (maximal accumulative) extensions of* $\mathcal{L}_{\min}$ *with separated boundary conditions. The self-adjoint extensions of* $\mathcal{L}_{\min}$ *are obtained precisely when* $\Im\alpha_1 = 0$ *or* $\alpha_1 = \infty$, *and* $\Im\alpha_2 = 0$ *or* $\alpha_2 = \infty$. *Here for* $\alpha_1 = \infty$ $(\alpha_2 = \infty)$ *condition (3.17) ((3.18)) should be replaced by* $x\,(0) = 0$ $([x,\psi]\,(\infty) = 0)$.

4. Conclusion

We consider the regular q-Sturm–Liouville expression (2.1) in $\mathbb{I} := [0,a]$, $0 < a < \infty$. In the Hilbert space $\mathcal{L}_q^2(\mathbb{I})$ consisting of all complex-valued functions x such that

$$\int_0^a |x(t)|^2\, d_q t < \infty$$

with the inner product

$$(x,y) = \int_0^a x(t)\overline{y(t)} d_q t,$$

we denote by $\mathcal{D}_{\max}$ the linear set of all functions $x \in \mathcal{L}_q^2(\mathbb{I})$ such that x and $D_q x$ are q-regular functions at 0 and $Lx \in \mathcal{L}_q^2\,(\mathbb{I})$. We define the maximal operator $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max} x = Lx$. In $\mathcal{L}_q^2\,(\mathbb{I})$, we consider the linear dense set $\mathcal{D}_{\min}$ consisting of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$x\,(0) = D_{q^{-1}} x\,(0) = 0,\ (x)(a) = D_{q^{-1}} x(a) = 0.$$

Denote by $\mathcal{L}_{\min}$ the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$.The minimal operator $\mathcal{L}_{\min}$ is a closed, symmetric operator with deficiency indices (2, 2), and $\mathcal{L}_{\max} = \mathcal{L}_{\min}^*$. We construct a space of boundary values of the minimal symmetric operator $\mathcal{L}_{\min}$, and describe all maximal dissipative, maximal accumulative, self-adjoint, dissipative, accumulative and symmetric extensions of symmetric q-Sturm–Liouville operator $\mathcal{L}_{\min}$ in terms of the boundary conditions at 0 and a. In particular, we describe all maximal dissipative, maximal accumulative, and self-adjoint extensions of $\mathcal{L}_{\min}$ with separated boundary conditions.

Further, we consider the singular q-Sturm–Liouville expression (3,1) in $\mathcal{I} := [0,\infty)$. In the Hilbert space $\mathcal{L}_q^2(\mathcal{I})$, denote by $\mathcal{D}_{\max}$ the linear set of all functions $x \in \mathcal{L}_q^2(\mathcal{I})$ such that x and $D_q x$ are q-regular functions at

0 and $Lx \in \mathcal{L}_q^2(\mathcal{I})$. We define the maximal operator $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max}x = Lx$. In the space $\mathcal{L}_q^2(\mathcal{I})$, we consider the linear dense set $\mathcal{D}_{\min}$ consisting of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$x(0) = D_{q^{-1}}x(0) = 0, \ \ [x, y](\infty) = 0, \ \forall y \in \mathcal{D}_{\max},$$

where

$$[x, y](\infty) = \lim_{t\to\infty}[x(t)\overline{D_{q^{-1}}y(t)} - D_{q^{-1}}x(t)\overline{y(t)}].$$

Denote by $\mathcal{L}_{\min}$ the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_{\min}$. The minimal operator $\mathcal{L}_{\min}$ is a closed symmetric operator with deficiency indices $(1, 1)$ or $(2, 2)$, and $\mathcal{L}_{\max} = \mathcal{L}^*_{\min}$. Let symmetric operator $\mathcal{L}_{\min}$ has deficiency indices $(1, 1)$, so the case of Weyl's limit-point occurs for the expression L or the operator $\mathcal{L}_{\min}$. Then $[x, y](\infty) = 0$ for all $x, y \in \mathcal{D}_{\min}$. The domain $\mathcal{D}_{\min}$ of the operator $\mathcal{L}_{\min}$ consist of vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions $x(0) = D_{q^{-1}}x(0) = 0$. In this case, we construct a space of boundary values of the symmetric q-Sturm–Liouville operator $\mathcal{L}_{\min}$, and describe all maximal dissipative, maximal accumulative and self-adjoint extensions of symmetric operator $\mathcal{L}_{\min}$ in terms of the boundary conditions at 0.

We assume that minimal symmetric q-Sturm–Liouville operator $\mathcal{L}_{\min}$ has deficiency indices $(2, 2)$, so that the Weyl's limit-circle case holds for the expression L or the operator $\mathcal{L}_{\min}$. We construct a space of boundary values of the symmetric operator $\mathcal{L}_{\min}$, and describe all maximal dissipative, maximal accumulative, self-adjoint, dissipative, accumulative and symmetric extensions of symmetric operator $\mathcal{L}_{\min}$ in terms of the boundary conditions at 0 and ∞. In particular, we describe all maximal dissipative, maximal accumulative and self-adjoint extensions of operator $\mathcal{L}_{\min}$ with separated boundary conditions at 0 and ∞.

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