

## On the Solution for a Nonlinear Singular $q$ -Sturm-Liouville Problems on the Whole Axis

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**Abstract.** In this paper, we consider a nonlinear  $q$ -Sturm–Liouville problem on the whole real axis in which the limit-circle case holds for  $q$ -Sturm–Liouville expression at infinity. We established the existence and uniqueness of solutions for this problem.

**Keywords:**  $q$ -Sturm–Liouville problem; Singular point; Weyl limit-circle case; Completely continuous operator; Fixed point theorems.

### 1. Introduction

In the last few decades, the  $q$ -difference equations have been studied extensively because it plays an important role in different mathematical and physical areas, such as the calculus of variations, mechanics, orthogonal polynomials, statistic physics, nuclear and high energy physics, conformal quantum mechanics, and theory of relativity. For a general introduction to the quantum calculus we refer the reader to the references [16, 11, 7].

Although much work has been done on the existence of solutions of  $q$ -difference equations (see [1, 2, 3, 4, 21, 22, 23, 25, 26, 10, 15, 14, 19, 20, 9]), no one

has studied the existence of solutions for singular impulsive nonlinear  $q$ -Sturm–Liouville problems that the limit-circle case holds at infinity. Our goal is to fill the gap in this area by using a special way to pose boundary conditions at infinity. In the analysis that follows, we will largely follow the development of the theory in [5, 6, 12].

In the following section, we introduce some necessary fundamental concepts of quantum calculus. We use the standard notations found in [16, 7].

## 2. Preliminaries

Let  $q$  be a positive number with  $0 < q < 1$ ,  $A \subset \mathbb{R} := (-\infty, \infty)$  and  $0 \in A$ . A  $q$ -difference equation is an equation that contains  $q$ -derivatives of a function defined on  $A$ . Let  $y$  be a complex-valued function on  $A$ . The  $q$ -difference operator  $D_q$ , the Jackson  $q$ -derivative is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{qx - x} \text{ for all } x \in A.$$

Note that there is a connection between  $q$ -deformed Heisenberg uncertainty relation and the Jackson derivative on  $q$ -basic numbers (see [24]). In the  $q$ -derivative, as  $q \rightarrow 1$ , the  $q$ -derivative is reduced to the classical derivative. The  $q$ -derivative at zero is defined by

$$D_q y(0) = \lim_{n \rightarrow \infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on  $x$ . Since the formulation of the extension problems requires the definition of  $D_{q^{-1}}$  in a same manner to be

$$D_{q^{-1}} f(x) := \begin{cases} \frac{f(x) - f(q^{-1}x)}{x - q^{-1}x} & \text{if } x \in A \setminus \{0\}, \\ D_q f(0) & \text{if } x = 0, \end{cases}$$

provided that  $D_q f(0)$  exists. Associated with this operator there is a non-symmetric formula for the  $q$ -differentiation of a product

$$D_q[f(x)g(x)] = g(x)D_q f(x) + f(qx)D_q g(x).$$

A right-inverse to  $D_q$ , the Jackson  $q$ -integration is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t \quad (a, b \in A).$$

The  $q$ -integration for a function is defined in [13] by the formulas

$$\begin{aligned} \int_0^\infty f(t) d_q t &= (1-q) \sum_{n=-\infty}^\infty q^n f(q^n), \\ \int_{-\infty}^0 f(t) d_q t &= (1-q) \sum_{n=-\infty}^\infty q^n f(-q^n), \\ \int_{-\infty}^\infty f(t) d_q t &= (1-q) \sum_{n=-\infty}^\infty q^n [f(q^n) + f(-q^n)]. \end{aligned}$$

A function  $f$  which is defined on  $A$ ,  $0 \in A$ , is said to be  $q$ -regular at zero if

$$\lim_{n \rightarrow \infty} f(xq^n) = f(0),$$

for every  $x \in A$ . Through the remainder of the paper, we deal only with functions  $q$ -regular at zero.

If  $f$  and  $g$  are  $q$ -regular at zero, then we have

$$\int_0^a g(t) D_q f(t) d_q t - \int_0^a f(qt) D_q g(t) d_q t = f(a)g(a) - f(0)g(0).$$

Let  $L_q^2(\mathbb{R})$  be the space of all real-valued functions defined on  $\mathbb{R}$  such that

$$\|f\| := \left( \int_{-\infty}^\infty f^2(x) d_q x \right)^{1/2} < \infty.$$

The space  $L_q^2(\mathbb{R})$  is a separable Hilbert space with the inner product (see [7, 8])

$$(f, g) := \int_{-\infty}^\infty f(x)g(x) d_q x, \quad f, g \in L_q^2(\mathbb{R})$$

The  $q$ -Wronskian of  $y(x)$  and  $z(x)$  is defined to be

$$W_q(y, z)(x) := y(x) D_q z(x) - z(x) D_q y(x), \quad x \in \mathbb{R}. \tag{1}$$

Let us consider the following nonlinear  $q$ -Sturm–Liouville equation

$$\Lambda y := -\frac{1}{q} D_{q^{-1}}(p(x) D_q y(x)) + r(x)y(x) = f(x, y(x)), \tag{2}$$

where  $p, r$  are real-valued functions defined on  $\mathbb{R}$  and continuous at zero,  $\frac{1}{p}, r \in L_{q,loc}^1(\mathbb{R})$  and  $y = y(x)$  is a desired solution.

Denote by  $\mathcal{D}$  the linear set of all functions  $y \in L_q^2(\mathbb{R})$  such that  $y$  and  $pD_q y$  are  $q$ -regular at zero and  $\Lambda(y) \in L_q^2(\mathbb{R})$ . The operator  $L$  defined by  $Ly = \Lambda(y)$  is called the *maximal operator* on  $L_q^2(\mathbb{R})$ .

For every  $y, z \in \mathcal{D}$  we have  $q$ -Green's formula (or  $q$ -Lagrange's identity)

$$\int_0^t (Ly)(x)z(x)d_qx - \int_0^t y(x)(Lz)(x)d_qx = [y, z]_t - [y, z]_0, \quad t \in \mathbb{R}, \quad (3)$$

where  $[y, z]_x := p(x)\{y(x)D_{q^{-1}}z(x) - D_{q^{-1}}y(x)z(x)\}$  (see [7, 8]).

It is clear from (3) that limit

$$[y, z]_{\pm\infty} = \lim_{n \rightarrow \infty} [y, z](\pm q^{-n})$$

exists and is finite for all  $y, z \in \mathcal{D}$ .

For any function  $y \in \mathcal{D}$ ,  $y(0)$  and  $(pD_{q^{-1}}y)(0)$  can be defined by

$$\begin{aligned} y(0) &:= \lim_{n \rightarrow \infty} y(q^n), \\ (pD_{q^{-1}}y)(0) &:= \lim_{n \rightarrow \infty} (pD_{q^{-1}}y)(q^n). \end{aligned}$$

These limits exist and are finite (since  $y$  and  $(pD_{q^{-1}})y$  are  $q$ -regular at zero).

We assume that the following conditions are satisfied:

(A1) The functions  $p$  and  $r$  are such that all solutions of the the equation

$$\Lambda(y) = 0 \quad (4)$$

belong to  $L_q^2(\mathbb{R})$ , i.e., Weyl limit-circle case holds for the  $q$ -Sturm–Liouville expression  $\Lambda$  [8].

(A2) The function  $f(x, \zeta)$  is a real-valued and continuous in  $(x, \zeta) \in \mathbb{R} \times \mathbb{R}$ , and

$$|f(x, \zeta)| \leq g(x) + \mu|\zeta| \quad (5)$$

for all  $(x, \zeta)$  in  $\mathbb{R} \times \mathbb{R}$ , where  $g(x) \geq 0$ ,  $g \in L_q^2(\mathbb{R})$ , and  $\mu$  is a positive constant.

Denote by  $u(x)$  and  $v(x)$  the solution of the equation (4) satisfying the initial conditions

$$u(0) = 0, \quad (pD_{q^{-1}}u)(0) = 1, \quad v(0) = -1, \quad (pD_{q^{-1}}v)(0) = 0. \quad (6)$$

Since the Wronskian of any two solutions of equation (4) are constant, we have  $W_q(u, v) = 1$ . Then,  $u$  and  $v$  are linearly independent and they form a fundamental system of solutions of equation (4). By the condition (A1), we get  $u, v \in L_q^2(\mathbb{R})$  and moreover,  $u, v \in \mathcal{D}$ . So, the values  $[y, u]_{\pm\infty}$  and  $[y, v]_{\pm\infty}$  exist and are finite for every  $y \in \mathcal{D}$ . By using Green's formula (3) and the conditions (6), we can get

$$\begin{aligned} [y, u]_{-\infty} &= y(0) - \int_{-\infty}^0 u(x)(\Lambda y)(x)d_qx, \\ [y, v]_{-\infty} &= (pD_{q^{-1}}y)(0) - \int_{-\infty}^0 v(x)(\Lambda y)(x)d_qx, \\ [y, u]_{\infty} &= y(0) + \int_{-\infty}^{\infty} u(x)(\Lambda y)(x)d_qx, \\ [y, v]_{\infty} &= (pD_{q^{-1}}y)(0) + \int_{-\infty}^{\infty} v(x)(\Lambda y)(x)d_qx. \end{aligned} \quad (7)$$

Now, we will add to problem (2) the boundary conditions

$$\begin{aligned} [y, u]_{-\infty} \cos \alpha + [y, v]_{-\infty} \sin \alpha &= c_1, \\ [y, u]_{\infty} \cos \beta + [y, v]_{\infty} \sin \beta &= c_2, \end{aligned} \tag{8}$$

where  $\alpha, \beta \in \mathbb{R}$ ,

(A3)  $\rho := \cos \alpha \sin \beta - \cos \beta \sin \alpha \neq 0$ ,

and  $c_1, c_2$  are arbitrary given real numbers.

Since the function  $y$  in (8) satisfies equation (2), we have

$$\begin{aligned} [y, u]_{-\infty} &= y(0) - \int_{-\infty}^0 u(x) f(x, y(x)) d_q x, \\ [y, v]_{-\infty} &= (pD_{q^{-1}}y)(0) - \int_{-\infty}^0 v(x) f(x, y(x)) d_q x, \\ [y, u]_{\infty} &= y(0) + \int_0^{\infty} u(x) f(x, y(x)) d_q x, \\ [y, v]_{\infty} &= (pD_{q^{-1}}y)(0) + \int_0^{\infty} v(x) f(x, y(x)) d_q x. \end{aligned}$$

### 3. Green’s Function

In this section, we consider the Green’s function for the boundary-value problem (2), (8). Then, we define a fixed point problem by using the Green’s function.

Consider the linear boundary value problem

$$-\frac{1}{q}D_{q^{-1}}(p(x)D_q y(x)) + r(x)y(x) = g(x), \quad x \in \mathbb{R}, \quad g \in L_q^2(\mathbb{R}) \tag{9}$$

$$\begin{aligned} [y, u]_{-\infty} \cos \alpha + [y, v]_{-\infty} \sin \alpha &= 0, \quad \alpha \in \mathbb{R}, \\ [y, u]_{\infty} \cos \beta + [y, v]_{\infty} \sin \beta &= 0, \quad \beta \in \mathbb{R}, \end{aligned} \tag{10}$$

where  $y$  is a desired solution,  $u$  and  $v$  are solutions of equation (4) under the conditions (6).

Set

$$\varphi(x) = \cos \alpha u(x) + \sin \alpha v(x), \quad \psi(x) = \cos \beta u(x) + \sin \beta v(x), \tag{11}$$

where  $W_q(\varphi, \psi) = \cos \alpha \sin \beta - \cos \beta \sin \alpha = W$ . It is clear that these functions are solutions of equation (4) and are in  $L_q^2(\mathbb{R})$ . Further, we have

$$[\varphi, u]_x = \varphi(0) = -\sin \alpha, \quad [\varphi, v]_x = (pD_{q^{-1}}\varphi)(0) = \cos \alpha, \tag{12}$$

$$[\psi, u]_x = \psi(0) = -\sin \beta, \quad [\psi, v]_x = (pD_{q^{-1}}\psi)(0) = \cos \beta, \tag{13}$$

$$\begin{aligned} [\varphi, u]_{\pm\infty} &= -\sin \alpha, \quad [\varphi, v]_{\pm\infty} = \cos \alpha, \\ [\psi, u]_{\pm\infty} &= -\sin \beta, \quad [\psi, v]_{\pm\infty} = \cos \beta. \end{aligned} \tag{14}$$

Define the Green's function of the boundary-value problem (9)-(10) by the formula

$$G(x, t) = \begin{cases} -\frac{\varphi(x)\psi(t)}{W} & \text{if } -\infty < t \leq x < \infty, \\ -\frac{\varphi(t)\psi(x)}{W} & \text{if } -\infty < x < t < \infty. \end{cases} \quad (15)$$

Since  $\varphi, \psi \in L_q^2(\mathbb{R})$ ,  $G(x, t)$  is a  $q$ -Hilbert-Schmidt kernel, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t < \infty. \quad (16)$$

**Theorem 3.1.** *The function*

$$y(x) = \int_{-\infty}^{\infty} G(x, t) g(t) d_q t, \quad x \in \mathbb{R}, \quad (17)$$

*is the unique solution of the boundary-value problem (9)-(10).*

*Proof.* Using the variation of constants formula, the general solution of equation (9) has the form

$$\begin{aligned} y(x) = & k_1 \varphi(x) + k_2 \psi(x) + \frac{q}{W} \psi(x) \int_{-\infty}^x \varphi(qt) g(qt) d_q t \\ & - \frac{q}{W} \varphi(x) \int_{-\infty}^x \psi(qt) g(qt) d_q t, \end{aligned} \quad (18)$$

where  $k_1$  and  $k_2$  are arbitrary constants.

By (18), we get

$$\begin{aligned} (pD_{q^{-1}}y)(x) = & k_1 (pD_{q^{-1}}\varphi)(x) + k_2 (pD_{q^{-1}}\psi)(x) \\ & + \frac{q}{W} (pD_{q^{-1}}\psi)(x) \int_{-\infty}^x \varphi(qt) g(qt) d_q t \\ & - \frac{q}{W} (pD_{q^{-1}}\varphi)(x) \int_{-\infty}^x \psi(qt) g(qt) d_q t. \end{aligned}$$

Hence, we have

$$\begin{aligned} [y, u]_x = & p(x) \{y(x)D_{q^{-1}}u(x) - D_{q^{-1}}y(x)u(x)\} \\ = & k_1 [\varphi, u]_x + k_2 [\psi, u]_x + \frac{q}{W} [\psi, u]_x \int_{-\infty}^x \varphi(qt) g(qt) d_q t \\ & - \frac{q}{W} [\varphi, u]_x \int_{-\infty}^x \psi(qt) g(qt) d_q t = -k_1 \sin \alpha - k_2 \sin \beta \\ & - \frac{q}{W} \sin \beta \int_{-\infty}^x \varphi(qt) g(qt) d_q t + \frac{q}{W} \sin \alpha \int_{-\infty}^x \psi(qt) g(qt) d_q t \\ = & -k_1 \sin \alpha - k_2 \sin \beta + \frac{q}{W} \int_{-\infty}^x (-\sin \beta \varphi(qt) + \sin \alpha \psi(qt)) g(qt) d_q t \\ = & -k_1 \sin \alpha - k_2 \sin \beta + \frac{q}{W} \int_{-\infty}^x u(qt) g(qt) d_q t. \end{aligned} \quad (19)$$

Likewise

$$\begin{aligned}
 [y, v]_x &= p(x) \{y(x)D_{q^{-1}}v(x) - D_{q^{-1}}y(x)v(x)\} \\
 &= k_1[\varphi, v]_x + k_2[\psi, v]_x + \frac{q}{W}[\psi, v]_x \int_{-\infty}^x \varphi(qt)g(qt)d_qt \\
 &\quad - \frac{q}{W}[\varphi, v]_x \int_{-\infty}^x \psi(qt)g(qt)d_qt = k_1 \cos \alpha + k_2 \cos \beta \\
 &\quad + \frac{q}{W} \cos \beta \int_{-\infty}^x \varphi(qt)g(qt)d_qt - \frac{q}{W} \cos \alpha \int_{-\infty}^x \psi(qt)g(qt)d_qt \tag{20} \\
 &= k_1 \cos \alpha + k_2 \cos \beta + \frac{q}{W} \int_{-\infty}^x (-\cos \beta \varphi(qt) + \cos \alpha \psi(qt))g(qt)d_qt \\
 &= k_1 \cos \alpha + k_2 \cos \beta + \frac{q}{W} \int_{-\infty}^x v(qt)g(qt)d_qt.
 \end{aligned}$$

From (19) and (20), we get

$$\begin{aligned}
 [y, u]_{-\infty} &= -k_1 \sin \alpha - k_2 \sin \beta, \\
 [y, v]_{-\infty} &= k_1 \cos \alpha + k_2 \cos \beta.
 \end{aligned} \tag{21}$$

Substituting (21) into (10), we obtain

$$k_2 (\cos \alpha \sin \beta - \sin \alpha \cos \beta) = 0, \quad k_2 W = 0,$$

i.e.,  $k_2 = 0$ . Further, we have

$$\begin{aligned}
 [y, u]_{\infty} &= -k_1 \sin \alpha + \frac{q}{W} \int_{-\infty}^{\infty} u(qt)g(qt)d_qt, \\
 [y, v]_{\infty} &= k_1 \cos \alpha + \frac{q}{W} \int_{-\infty}^{\infty} v(qt)g(qt)d_qt.
 \end{aligned}$$

From the conditions (10), we have

$$k_1 (-\sin \alpha \cos \beta + \cos \alpha \sin \beta) + q \int_{-\infty}^{\infty} [\cos \beta u(qt) + \sin \beta v(qt)]g(qt)d_qt = 0.$$

Hence,

$$k_1 = -\frac{q}{W} \int_{-\infty}^{\infty} \psi(qt)g(qt)d_qt.$$

By (18), we get

$$y(x) = -\frac{q}{W} \int_{-\infty}^x \varphi(qt)\psi(x)g(qt)d_qt - \frac{q}{W} \int_x^{\infty} \varphi(x)\psi(qt)g(qt)d_qt$$

i.e., (15) and (17) hold. ■

**Theorem 3.2.** *The unique solution of the equation (9) under the conditions (8) is given by the formula*

$$y(x) = \omega(x) + \int_{-\infty}^{\infty} G(x, t) g(t) d_q t,$$

where

$$\omega(x) = \frac{c_2}{W} \varphi(x) - \frac{c_1}{W} \psi(x).$$

*Proof.* By the conditions (12)-(14), the function  $\omega(x)$  is a unique solution of the equation (4) satisfying the conditions (8). By Theorem 3.1 the function  $(G(x, \cdot), g(\cdot))$  a unique solution of the equation (9) satisfying the conditions (10). This completes the proof. ■

From Theorem 3.2, the boundary-value problem (2), (8) in  $L_q^2(\mathbb{R})$  is equivalent to the nonlinear  $q$ -integral equation

$$y(x) = \omega(x) + \int_{-\infty}^{\infty} G(x, t) f(t, y(t)) d_q t, \quad (22)$$

where the functions  $\omega(x)$  and  $G(x, t)$  are defined above. Hence, we shall study the equation (22).

By (5) and (16), we can define the operator  $T : L_q^2(\mathbb{R}) \rightarrow L_q^2(\mathbb{R})$  by the formula

$$(Ty)(x) = \omega(x) + \int_{-\infty}^{\infty} G(x, t) f(t, y(t)) d_q t, \quad x \in \mathbb{R}, \quad (23)$$

where  $y, \omega \in L_q^2(\mathbb{R})$ . Then the equation (22) can be written as  $y = Ty$ .

Now, our next goal is to search the fixed points of the operator  $T$  because it is equivalent to solving the equation (22).

#### 4. The Fixed Points of the Operator T

In this section, we investigate the fixed points of the operator  $T$  by using the following Banach fixed point theorem.

**Definition 4.1.** [17] *Let  $A$  be a mapping of a metric space  $R$  into itself. Then  $x$  is called a fixed point of  $A$  if  $Ax = x$ . Suppose there exists a number  $\alpha < 1$  such that*

$$\rho(Ax, Ay) \leq \alpha \rho(x, y)$$

for every pair of points  $x, y \in R$ . Then  $A$  is said to be a contraction mapping.

**Theorem 4.2.** [17] *Every contraction mapping  $A$  defined on a complete metric space  $R$  has a unique fixed point.*



**Theorem 4.3.** *Suppose that the conditions (A1), (A2) and (A3) are satisfied. Further, let the function  $f(x, y)$  satisfy the following Lipschitz condition: there exist a constant  $K > 0$  such that*

$$\int_{-\infty}^{\infty} |f(x, y(x)) - f(x, z(x))|^2 d_q x \leq K^2 \int_{-\infty}^{\infty} |y(x) - z(x)|^2 d_q x \quad (24)$$

for all  $y, z \in L_q^2(\mathbb{R})$ . If

$$K \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t \right)^{1/2} < 1, \quad (25)$$

then the boundary-value problem (2), (8) has a unique solution in  $L_q^2(\mathbb{R})$ .

*Proof.* It is sufficient to show that the operator  $T$  is a contraction operator. For  $y, z \in L_q^2(\mathbb{R})$ , we have

$$\begin{aligned} & |(Ty)(x) - (Tz)(x)|^2 \\ &= \left| \int_{-\infty}^{\infty} G(x, t) [f(t, y(t)) - f(t, z(t))] d_q t \right|^2 \\ &\leq \int_{-\infty}^{\infty} |G(x, t)|^2 d_q t \int_{-\infty}^{\infty} |f(t, y(t)) - f(t, z(t))|^2 d_q t \\ &\leq K^2 \|y - z\|^2 \int_{-\infty}^{\infty} |G(x, t)|^2 d_q t, \quad x \in \mathbb{R}. \end{aligned}$$

Thus, we get

$$\|Ty - Tz\| \leq \alpha \|y - z\|,$$

where

$$\alpha = K \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t \right)^{1/2} < 1,$$

i.e.,  $T$  is a contraction mapping. ■

Now, we claim that the function  $f(x, y)$  satisfies a Lipschitz condition on a subset of  $L_q^2(\mathbb{R})$  but not of the whole space.

**Theorem 4.4.** *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function  $f(x, y)$  satisfy the following Lipschitz condition: there exist constants  $M, K > 0$  such that*

$$\int_{-\infty}^{\infty} |f(x, y(x)) - f(x, z(x))|^2 d_q x \leq K^2 \int_0^{\infty} |y(x) - z(x)|^2 d_q x \quad (26)$$

for all  $y$  and  $z$  in  $S_M = \{y \in L_q^2(\mathbb{R}) : \|y\| \leq M\}$ , where  $K$  may depend on  $M$ .  
If

$$\begin{aligned} & \left\{ \int_{-\infty}^{\infty} |\omega(x)|^2 d_q x \right\}^{1/2} + \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x,t)|^2 d_q x d_q t \right)^{1/2} \\ & \times \sup_{y \in S_M} \left\{ \int_{-\infty}^{\infty} |f(t, y(t))|^2 d_q t \right\}^{1/2} \leq M \end{aligned} \quad (27)$$

and

$$K \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x,t)|^2 d_q x d_q t \right)^{1/2} < 1, \quad (28)$$

then the boundary-value problem (2), (8) has a unique solution with

$$\int_{-\infty}^{\infty} |y(x)|^2 d_q x \leq M^2.$$

*Proof.* It is clear that  $S_M$  is a closed set of  $L_q^2(\mathbb{R})$ . Firstly, we will prove that the operator  $T$  maps  $S_M$  into itself. For  $y \in S_M$  we have

$$\begin{aligned} \|Ty\| &= \left\| \omega(\cdot) + \int_{-\infty}^{\infty} G(\cdot, t) f(t, y(t)) d_q t \right\| \\ &\leq \|\omega\| + \left\| \int_{-\infty}^{\infty} G(\cdot, t) f(t, y(t)) d_q t \right\| \\ &\leq \|\omega\| + \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x,t)|^2 d_q x d_q t \right)^{1/2} \\ &\quad \times \sup_{y \in S_M} \left\{ \int_{-\infty}^{\infty} |f(t, y(t))|^2 d_q t \right\}^{1/2} \leq M. \end{aligned}$$

Thus,  $T : S_M \rightarrow S_M$ .

We now proceed analogously to the proof of Theorem 4.3. So, we can get

$$\|Ty - Tz\| \leq \alpha \|y - z\|, \quad y, z \in S_M, \quad \alpha < 1.$$

If we apply the Banach fixed point theorem, then we obtain a unique solution of the boundary-value problem (2), (8) in  $S_M$ .  $\blacksquare$

## 5. An Existence Theorem Without Uniqueness

In this section, we shall prove an existence theorem without uniqueness of solution, using the following Schauder fixed point theorem:

**Definition 5.1.** [12] *An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

**Theorem 5.2.** [12] *Let  $\mathbf{B}$  be a Banach space and  $\mathbf{S}$  a nonempty bounded, convex, and closed subset of  $\mathbf{B}$ . Assume that  $A : \mathbf{B} \rightarrow \mathbf{B}$  is a completely continuous operator. If the operator  $A$  leaves the set  $\mathbf{S}$  invariant, i.e., if  $A(\mathbf{S}) \subset \mathbf{S}$ , then  $A$  has at least one fixed point in  $\mathbf{S}$ .*

**Theorem 5.3.** *The operator  $T$  defined by (23) is completely continuous operator under the conditions (A1), (A2) and (A3).*

*Proof.* Let  $y_0 \in L_q^2(\mathbb{R})$ . Then, we obtain

$$\begin{aligned} & |(Ty)(x) - (Ty_0)(x)|^2 \\ &= \left| \int_{-\infty}^{\infty} G(x,t) [f(t,y(t)) - f(t,y_0(t))] d_q t \right|^2 \\ &\leq \int_{-\infty}^{\infty} |G(x,t)|^2 d_q t \int_{-\infty}^{\infty} |f(t,y(t)) - f(t,y_0(t))|^2 d_q t. \end{aligned}$$

Thus

$$\|Ty - Ty_0\|^2 \leq K \int_{-\infty}^{\infty} |f(t,y(t)) - f(t,y_0(t))|^2 d_q t, \tag{29}$$

where

$$K = \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x,t)|^2 d_q x d_q t \right).$$

We know that an operator  $F$  defined by  $Fy(x) = f(x,y(x))$  is continuous in  $L_q^2(\mathbb{R})$  under the condition (A2) ( see [18]). Hence, for the given  $\epsilon > 0$ , we can find a  $\delta > 0$  such that  $\|y - y_0\| < \delta$  implies

$$\int_{-\infty}^{\infty} |f(t,y(t)) - f(t,y_0(t))|^2 d_q t < \frac{\epsilon^2}{K}.$$

From (29), we get

$$\|Ty - Ty_0\| < \epsilon,$$

i.e.,  $T$  is continuous.

Set  $Y = \{y \in L_q^2(\mathbb{R}) : \|y\| \leq C\}$ . By (23), we have

$$\|Ty\| \leq \|\omega\| + \left\{ K \int_{-\infty}^{\infty} |f(t,y(t))|^2 d_q t \right\}^{1/2},$$

for all  $y \in Y$ . Furthermore, using (5), we get

$$\begin{aligned} & \int_{-\infty}^{\infty} |f(t,y(t))|^2 d_q t \leq \int_{-\infty}^{\infty} [g(t) + \mu |y(t)|]^2 d_q t \\ & \leq 2 \int_{-\infty}^{\infty} [g^2(t) + \mu^2 |y(t)|^2] d_q t = 2 \left( \|g\|^2 + \mu^2 \|y\|^2 \right) \\ & \leq 2(\|g\|^2 + \mu^2 C^2). \end{aligned}$$

Thus, for all  $y \in Y$ , we obtain

$$\|Ty\| \leq \|\omega\| + \left[2K \left(\|g\|^2 + \mu^2 C^2\right)\right]^{1/2},$$

i.e.,  $T(Y)$  is a bounded set in  $L_q^2(\mathbb{R})$ .

Further, for all  $y \in Y$ , we have

$$\begin{aligned} & \int_{-\infty}^N |Ty(x)|^2 d_q x + \int_N^{\infty} |Ty(x)|^2 d_q x \\ & \leq 2(\|g\|^2 + \mu^2 C^2) \left\{ \int_{-\infty}^N \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t + \int_N^{\infty} \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t \right\}. \end{aligned}$$

So, from (16), we see that for given  $\epsilon > 0$  there exists a positive number  $N$ , depending only on  $\epsilon$  such that

$$\int_{-\infty}^N |Ty(x)|^2 d_q x + \int_N^{\infty} |Ty(x)|^2 d_q x < \epsilon^2,$$

for all  $y \in Y$ .

Thus  $T(Y)$  is a relatively compact in  $L_q^2(\mathbb{R})$ , i.e., the operator  $T$  is completely continuous. ■

**Theorem 5.4.** *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, there exist constants  $M > 0$  such that*

$$\begin{aligned} & \left\{ \int_{-\infty}^{\infty} |\omega(x)|^2 d_q x \right\}^{1/2} + \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(x, t)|^2 d_q x d_q t \right)^{1/2} \\ & \times \sup_{y \in S_M} \left\{ \int_{-\infty}^{\infty} |f(t, y(t))|^2 d_q t \right\}^{1/2} \leq M, \end{aligned} \quad (30)$$

where  $S_M = \{y \in L_q^2(\mathbb{R}) : \|y\| \leq M\}$ . Then the boundary-value problem (2), (8) has at least one solution with

$$\int_{-\infty}^{\infty} |y(x)|^2 d_q x \leq M^2.$$

*Proof.* Let us define an operator  $T : L_q^2(\mathbb{R}) \rightarrow L_q^2(\mathbb{R})$  by (23). From Theorems 4.4, 5.3 and (30), we conclude that  $T$  maps the set  $S_M$  into itself. It is clear that the set  $S_M$  is bounded, convex and closed. Using by Theorem 5.2, the theorem follows. ■

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