LE MATEMATICHE
Vol. LXXVII (2022) – Issue II, pp. 389–405
doi: 10.4418/2022.77.2.7

HAHN MULTIPLICATIVE CALCULUS

B. P. ALLAHVERDIEV - H. TUNA

In this study, Hahn multiplicative calculus was introduced. Some basic theorems are proved within this calculus. As an application of this subject, the classical Sturm–Liouville problem was examined under this structure.

1. Introduction

In 1949, Hahn combined two well-known operators under one definition ([12]). With this definition he made, the quantum q-difference operator (see [13]) and the forward difference operator (see [14]) were examined under a single structure. Until recently, this definition did not receive much attention. With the definition of the right inverse of the Hahn derivative operator, the Hahn calculus theory expanded and attracted the attention of researchers (see [1, 3, 4]).

Meanwhile, in the 1970s, Grossman and Katz introduced the Non-Newtonian calculus (see [9, 10]). They made a new definition of derivatives and integrals and turned addition and subtraction into multiplication and division. Although it did not attract the attention of researchers until the 2000s, interest in this subject has increased recently (see [2, 5–8, 11, 15]). In 2016, Yener and Emiroğlu

Received on July 19, 2022

AMS 2010 Subject Classification: 39A13, 11D57, 34B24
Keywords: Hahn multiplicative derivative, Hahn multiplicative integral, Sturm–Liouville problem

introduced the concept of multiplicative calculus for quantum calculus ([16]). However, we are not aware of any result related to the Hahn multiplicative calculus.

In this study, we will generalize the work of Yener and Emiroğlu for Hahn calculus and introduce Hahn multiplicative calculus.

2. Hahn multiplicative derivative

Throughout the paper, we let $q \in (0,1)$ and $\omega > 0$. Define $\omega_0 := \omega/(1-q)$ and let I be a real interval containing ω_0. Now let's remember the concept of the Hahn derivative

Definition 2.1 ([12]). Let $y : I \to \mathbb{R}$ be a function. The Hahn derivative operator $D_{\omega,q}y$ is given by

$$D_{\omega,q}y(x) = \begin{cases} [y(\omega+qx) - y(x)] \frac{1}{\omega+(q-1)x}, & x \neq \omega_0, \\ y'(\omega_0), & x = \omega_0, \end{cases}$$

provided that y is differentiable at ω_0.

The Hahn derivative operator has the following features.

Theorem 2.2 ([3]). *Let $y, z : I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$, then we have for all $x \in I$:*

$$D_{\omega,q}(\gamma y + \delta z)(x) = \gamma D_{\omega,q}y(x) + \delta D_{\omega,q}z(x), \ \gamma, \delta \in I,$$

$$D_{\omega,q}(yz)(x) = D_{\omega,q}(y(x))z(x) + y(\omega + xq) D_{\omega,q}z(x),$$

$$D_{\omega,q}\left(\tfrac{y}{z}\right)(x) = [D_{\omega,q}(y(x))z(x) - y(x) D_{\omega,q}z(x)][z(x)z(\omega+xq)]^{-1}.$$

Now, we will define the Hahn multiplicative derivative.

Definition 2.3. Let y be a positive function. The Hahn multiplicative derivative $D^*_{\omega,q}$ is defined by

$$D^*_{\omega,q}y(x) = \left(\frac{y(\omega+qx)}{y(x)}\right)^{\frac{1}{\omega+(q-1)x}}.$$

Theorem 2.4. *Let y be a ω, q-differentiable function. Then we have*

$$D^*_{\omega,q}y(x) = e^{D_{\omega,q}(\ln y(x))}.$$

Proof. From Definition 2.3, we obtain

$$D^*_{\omega,q}y(x) = \left(\frac{y(\omega+qx)}{y(x)}\right)^{\frac{1}{\omega+(q-1)x}} = e^{\ln\left(\frac{y(\omega+qx)}{y(x)}\right)^{\frac{1}{\omega+(q-1)x}}}$$

$$= e^{\frac{1}{\omega+(q-1)x}\ln\left(\frac{y(\omega+qx)}{y(x)}\right)} = e^{\frac{\ln y(\omega+qx)-\ln y(x)}{\omega+(q-1)x}} = e^{D_{\omega,q}(\ln y(x))}.$$

□

Now let's get the basic properties of the Hahn multiplicative derivative.

Theorem 2.5. *Let y,z be $(\omega,q)^*$-differentiable functions. Then we have the following properties.*
i)

$$D^*_{\omega,q}(y(x)z(x)) = D^*_{\omega,q}(y(x))\,D^*_{\omega,q}(z(x)),$$

ii)

$$D^*_{\omega,q}\left(\frac{y(x)}{z(x)}\right) = \frac{D^*_{\omega,q}(y(x))}{D^*_{\omega,q}(z(x))},$$

iii)

$$D^*_{\omega,q}(cy(x)) = D^*_{\omega,q}(y(x)),$$

where c is a positive constant,
iv)

$$D^*_{\omega,q}\left(y(x)^{z(x)}\right) = \left(D^*_{\omega,q}y(x)\right)^{z(qx+\omega)} y(x)^{D_{\omega,q}g(x)},$$

v)

$$D^*_{\omega,q}(y\circ z)(x) = D^*_{\omega,q,z}(y(z(x)))^{D_{\omega,q}z(x)},$$

vi) for all constant functions $y(x)=c$, we have

$$D^*_{\omega,q}y(x) = 1,$$

vii)

$$\left[D^*_{-\omega q^{-1},q^{-1}}y(x)\right]^{1/q} = D^*_{\omega,q}y\left(h^{-1}(x)\right),$$

where $h(x) = qx+\omega$.

Proof. *i*)

$$D^*_{\omega,q}\left(y(x)z(x)\right)=e^{D_{\omega,q}(\ln(y(x)z(x)))}=e^{D_{\omega,q}(\ln(y(x)+\ln z(x)))}$$

$$=e^{D_{\omega,q}\ln(y(x))}.e^{D_{\omega,q}\ln(z(x))}$$

$$=D^*_{\omega,q}\left(y(x)\right)D^*_{\omega,q}\left(z(x)\right).$$

ii)

$$D^*_{\omega,q}\left(\frac{y(x)}{z(x)}\right)=e^{D_{\omega,q}\left(\ln\frac{y(x)}{z(x)}\right)}=e^{D_{\omega,q}\ln(y(x)-\ln z(x))}$$

$$=\frac{e^{D_{\omega,q}\ln(y(x))}}{e^{D_{\omega,q}\ln(z(x))}}=\frac{D^*_{\omega,q}\left(y(x)\right)}{D^*_{\omega,q}\left(z(x)\right)}.$$

iii)

$$D^*_{\omega,q}\left(cy(x)\right)=\left(\frac{cy(\omega+qx)}{cy(x)}\right)^{\frac{1}{\omega+(q-1)x}}$$

$$=\left(\frac{y(\omega+qx)}{y(x)}\right)^{\frac{1}{\omega+(q-1)x}}$$

$$=D^*_{\omega,q}\left(y(x)\right).$$

iv)

$$D^*_{\omega,q}\left(y(x)^{z(x)}\right)=e^{D_{\omega,q}\left(\ln\left(y(x)^{z(x)}\right)\right)}=e^{D_{\omega,q}(z(x)\ln(y(x)))}$$

$$=e^{z(\omega+xq)D_{\omega,q}(\ln(y(x)))+\ln(y(x))D_{\omega,q}z(x)}$$

$$=e^{z(\omega+xq)D_{\omega,q}(\ln(y(x)))}.e^{\ln(y(x))D_{\omega,q}z(x)}$$

$$=\left(D^*_{\omega,q}y(x)\right)^{z(\omega+qx)}y(x)^{D_{\omega,q}g(x)}.$$

v)

$$D^*_{\omega,q}(y\circ z)(x) = \left(\frac{y(z(\omega+qx))}{y(z(x))}\right)^{\frac{1}{\omega+(q-1)x}}$$

$$= \left(\frac{y(z(\omega+qx))}{y(z(x))}\right)^{\frac{1}{z(\omega+qx)-z(x)}\frac{z(\omega+qx)-z(x)}{[\omega+(q-1)x]}}$$
$$= D^*_{\omega,q,z}(y(z(x)))^{D_{\omega,q}g(x)}.$$

vi)

$$D^*_{\omega,q}y(x) = \left(\frac{c}{c}\right)^{\frac{1}{\omega+(q-1)x}} = 1.$$

vii)

$$D^*_{\omega,q}y\left(h^{-1}(x)\right) = e^{D_{\omega,q}\ln y\left(h^{-1}(x)\right)} = e^{\frac{\ln y(x)-\ln y\left(h^{-1}(x)\right)}{\omega+(q-1)x}}$$

$$= \left[\frac{e^{\ln y(x)}}{e^{\ln y(h^{-1}(x))}}\right]^{\frac{1}{\omega+(q-1)x}}$$

$$= \left(\left[\frac{y(x)}{y(h^{-1}(x))}\right]^{\frac{1}{-\omega q^{-1}+\left(q^{-1}-1\right)x}}\right)^{-1/q}$$

$$= \left(D^*_{-\omega q^{-1},q^{-1}}y(x)\right)^{1/q}.$$

□

Let us now define the higher-order Hahn multiplicative derivative.

Definition 2.6. Let y be a positive function. The nth order Hahn multiplicative derivative is defined by

$$D^{*(n)}_{\omega,q}y(x) = e^{D^{(n)}_{\omega,q}(\ln y(x))}.$$

3. Hahn multiplicative integral

Let's start this section by recalling the Jackson–Nörlund Integral definition.

Definition 3.1 (Jackson–Nörlund Integral [3])**.** Let $y : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define ω, q-integral of the function y by

$$\int_a^b y(x)\, d_{\omega,q}x := \int_{\omega_0}^b y(x)\, d_{\omega,q}x - \int_{\omega_0}^a y(x)\, d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x y(t)\, d_{\omega,q}t := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n y\left(\omega \frac{1-q^n}{1-q} + xq^n\right),\ x \in I$$

provided that the series converges at $x = a$ and $x = b$. In this case, y is called ω, q-integrable on $[a, b]$.

Definition 3.2. Let y be a function on I and $a, b, \omega_0 \in I$. Then we define the Hahn multiplicative integral ($(\omega, q)^*$-integral) of the function y by

$$\int_a^b y(x)^{d_{\omega,q}x} := e^{\int_a^b \ln y(x) d_{\omega,q}x}$$

$$= e^{\int_{\omega_0}^b \ln y(x) d_{\omega,q}x - \int_{\omega_0}^a \ln y(x) d_{\omega,q}x}.$$

It is easy to check that the Hahn multiplicative integral has the following properties.

Theorem 3.3. *Let y, z be $(\omega, q)^*$-integrable functions on I and $a, b, c \in I$. Then*
i)

$$\int_a^b \left(y(x)^k\right)^{d_{\omega,q}x} = \left(\int_a^b y(x)^{d_{\omega,q}x}\right)^k, \text{ where } k \in \mathbb{R},$$

ii)

$$\int_a^b (y(x)\, z(x))^{d_{\omega,q}x} = \left(\int_a^b y(x)^{d_{\omega,q}x}\right)\left(\int_a^b z(x)^{d_{\omega,q}x}\right),$$

iii)

$$\int_a^b \left(\frac{y(x)}{z(x)}\right)^{d_{\omega,q}x} = \frac{\int_a^b y(x)^{d_{\omega,q}x}}{\int_a^b z(x)^{d_{\omega,q}x}},$$

iv)

$$\int_a^b y(x)^{d_{\omega,q}x} = \int_a^c y(x)^{d_{\omega,q}x} \int_c^b y(x)^{d_{\omega,q}x}, \text{ where } a \le c \le b.$$

Now, the $(\omega,q)^{*}$-integration by parts will be proved.

Theorem 3.4. *Let y be $(\omega,q)^{*}$-integrable and z be ω,q-differentiable, they are continuous on the interval $\omega_0 \le a \le b$, then*

$$\int_a^b \left[\left(D^{*}_{\omega,q}y(t)\right)^{z(t)}\right]^{d\omega,qt} = \frac{y(b)^{z(b)}}{y(a)^{z(a)}}\left(\left[\int_a^b (y(qt+\omega))^{D_{\omega,q}z(t)}\right]^{d_{\omega,q}t}\right)^{-1}.$$

Proof. From Definition 3.2, we obtain

$$\int_a^b \left[\left(D^{*}_{\omega,q}y(t)\right)^{z(t)}\right]^{d_{\omega,q}t} = \int_a^b \left[e^{z(t)D_{\omega,q}(\ln y(t))}\right]^{d_{\omega,q}t} = e^{\int_a^b z(t)D_{\omega,q}(\ln y(t))d_{\omega,q}t}. \quad (1)$$

By using the ω,q-integration by parts ([3]), we have

$$\int_a^b z(t)\,D_{\omega,q}(\ln y(t))\,d_{\omega,q} = z(b)\ln y(b) - z(a)\ln y(a)$$

$$-\int_a^b \ln y(qt+\omega)\,D_{\omega,q}z(t)\,d_{\omega,q}. \quad (2)$$

It follows from (2) and Definition 3.2 that

$$\int_a^b \left[\left(D^{*}_{\omega,q}y(t)\right)^{z(t)}\right]^{d_{\omega,q}t} = \frac{e^{z(b)\ln y(b)}}{e^{z(a)\ln y(a)}}.\frac{1}{e^{\int_a^b \ln y(qt+\omega)D_{\omega,q}z(t)d_{\omega,q}}}$$

$$= \frac{y(b)^{z(b)}}{y(a)^{z(a)}}.\left(\left[\int_a^b (y(qt+\omega))^{D_{\omega,q}z(t)}\right]^{d_{\omega,q}t}\right)^{-1}.$$

□

4. An application: Sturm–Liouville problem

It is well-known that the Sturm–Liouville theory is important in many areas of sciences, physics and engineering. Therefore, in this section, let's examine the classical Sturm–Liouville problem as an application of Hahn multiplicative calculus.

Firstly, we will give the following notation we will use in this section. Let

$$y \oplus z = y.z,\ y \ominus z = \frac{y}{z},\ y \odot z = y^{\ln z} = z^{\ln y},$$

where $y,z \in \mathbb{R}^{+}$.

Definition 4.1 ([11]). Let $\mathcal{H} \neq \varnothing$ and $\langle .,. \rangle_* : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^+$ be a function such that the following axioms are satisfied for all $x, y, z \in \mathcal{H}$:
i)
$$\langle x \oplus y, z \rangle_* = \langle x, y \rangle_* \oplus \langle y, z \rangle_*,$$

ii)
$$\langle x, y \rangle_* = \langle y, x \rangle_*,$$
iii)
$$\langle x, x \rangle_* = 1 \text{ if and only if } x = 1,$$
iv)
$$\langle x, x \rangle_* \geq 1,$$
v)
$$\langle e^k \odot x, y \rangle_* = e^k \odot \langle x, y \rangle_*, \ k \in \mathbb{R}.$$
Then $(\mathcal{H}, \langle .,. \rangle_*)$ is called multiplicative inner product space.

Let
$$L^2_{*,\omega,q}(\omega_0, b) := \left\{ y : \int_{\omega_0}^{b} |y(x) \odot y(x)|^{d_{\omega,q}x} < \infty \right\}.$$
Then $L^2_{*,\omega,q}(\omega_0, b)$ is a multiplicative inner product space with
$$\langle .,. \rangle_{*,\omega,q} : L^2_{*,\omega,q}(\omega_0, b) \times L^2_{*,\omega,q}(\omega_0, b) \to \mathbb{R}^+,$$

$$\langle y, z \rangle_{*,\omega,q} = \int_{\omega_0}^{b} |y(x) \odot z(x)|^{d_{\omega,q}x}, \tag{3}$$
where $y, z \in L^2_{*,\omega,q}(\omega_0, b)$ are positive functions.

Now consider a boundary value problem which consists of
1. a Hahn multiplicative Sturm–Liouville (HMSL) equation of the form
$$\begin{aligned} \Phi(z) &:= \left(\left(D^*_{-\omega q^{-1}, q^{-1}} \right)^{1/q} D^*_{\omega,q} z(x) \right) \oplus \left(e^{r(x)} \odot z(x) \right) \\ &= e^{\lambda} \odot z(x), \ x \in [\omega_0, b], \end{aligned} \tag{4}$$
where $r(.)$ is a real-valued continuous function at ω_0 defined on $[\omega_0, b]$, and λ is a parameter independent of x; and
2. two supplementary conditions
$$\left(e^{\cos\gamma} \odot z(\omega_0)\right) \oplus \left(e^{\sin\gamma} \odot D^*_{\omega,q} z(\omega_0) \right) = 1, \tag{5}$$
$$\left(e^{\cos\delta} \odot z(b) \right) \oplus \left(e^{\sin\delta} \odot D^*_{\omega,q} z\left(h^{-1}(b)\right) \right) = 1, \tag{6}$$
where $\gamma, \delta \in \mathbb{R}$.

Theorem 4.2. *HMSL operator defined by (4)-(6) is formally self-adjoint on the space* $L^2_{*,\omega,q}(\omega_0,b)$.

Proof. Let $z,t \in L^2_{*,\omega,q}(\omega_0,b)$. From (3), we obtain

$$\langle \Phi z,t\rangle_{*,\omega,q} = \int_{\omega_0}^{b}\left|\left(\left[\left(D^{*}_{-\omega q^{-1},q^{-1}}\right)^{1/q} D^{*}_{\omega,q} z(x)\right]\left[z(x)^{r(x)}\right]\right)^{\ln t(x)}\right|^{d_{\omega,q}x}$$

$$= \int_{\omega_0}^{b}\left|\left[\left(D^{*}_{-\omega q^{-1},q^{-1}}\right)^{1/q} D^{*}_{\omega,q} z(x)\right]^{\ln t(x)}\right|^{d_{\omega,q}x}$$

$$\times \int_{\omega_0}^{a}\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_{\omega,q}x}.$$

By Theorem 2.5, we get

$$\langle \Phi z,t\rangle_{*,\omega,q} = \int_{\omega_0}^{b}\left|\left[D^{*}_{\omega,q}\left(D^{*}_{\omega,q} z\left(h^{-1}(x)\right)\right)\right]^{\ln t(x)}\right| \int_{\omega_0}^{b}\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_{\omega,q}x}.$$

From Theorem 3.4, we deduce that

$$\langle \Phi z,t\rangle_{*,\omega,q} = \frac{\left(D^{*}_{\omega,q} z\left(h^{-1}(b)\right)\right)^{\ln t(b)}}{\left(D^{*}_{\omega,q} z(\omega_0)\right)^{\ln t(\omega_0)}}$$

$$\times \frac{1}{\int_{\omega_0}^{b}\left|\left(D^{*}_{\omega,q} z(x)\right)^{D_{\omega,q}\ln t(x)}\right|^{d_{\omega,q}x}} \int_{\omega_0}^{b}\left|\left(z(x)^{r(x)}\right)^{\ln t}\right|^{d_{\omega,q}x}$$

$$= \frac{\left(D^{*}_{\omega,q} z\left(h^{-1}(b)\right)\right)^{\ln t(b)}}{\left(D^{*}_{\omega,q} z(\omega_0)\right)^{\ln t(\omega_0)}}$$

$$\times \frac{1}{e^{\int_{\omega_0}^{b} D_{\omega,q}\ln z(x) D_{\omega,q}\ln t(x) d_{\omega,q}x}} \int_{\omega_0}^{b}\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_{\omega,q}x}. \tag{7}$$

Similarly,

$$\langle z, \Phi t\rangle_{*,\omega,q} = \int_{\omega_0}^{b} \left| z(x)^{\ln\left[D^{*}_{\omega,q}\left(D^{*}_{\omega,q} t\left(h^{-1}(x)\right)\right)\right]} \right|^{d_{\omega,q}x}$$

$$\times \int_{\omega_0}^{b} \left| (z(x))^{\ln(t(x))^{r(x)}} \right|^{d_{\omega,q}x}$$

$$= \int_{\omega_0}^{b} \left| \left[D^{*}_{\omega,q}\left(D^{*}_{\omega,q} t\left(h^{-1}(x)\right)\right)\right]^{\ln z(x)} \right|^{d_{\omega,q}x}$$

$$\times \int_{\omega_0}^{b} \left| \left(t(x)^{r(x)}\right)^{\ln z(x)} \right|^{d_{\omega,q}x}$$

$$= \frac{\left(D^{*}_{\omega,q} t\left(h^{-1}(b)\right)\right)^{\ln z(b)}}{\left(D^{*}_{\omega,q} t(\omega_0)\right)^{\ln z(\omega_0)}}$$

$$\times \frac{1}{e^{\int_{\omega_0}^{b} D_{\omega,q}\ln t(x) D_{\omega,q}\ln z(x) d_{\omega,q}x}}$$

$$\times \int_{\omega_0}^{b} \left| \left(t(x)^{r(x)}\right)^{\ln z(x)} \right|^{d_{\omega,q}x}. \tag{8}$$

It follows from (7) and (8) that

$$\langle \Phi z, t\rangle_{*,\omega,q} = \frac{\dfrac{\left(D^{*}_{\omega,q} z\left(h^{-1}(b)\right)\right)^{\ln t(b)}}{\left(D^{*}_{\omega,q} z(\omega_0)\right)^{\ln t(\omega_0)}}}{\dfrac{\left(D^{*}_{\omega,q} t(h^{-1}(b))\right)^{\ln z(b)}}{\left(D^{*}_{\omega,q} t(\omega_0)\right)^{\ln z(\omega_0)}}} \langle z, \Phi t\rangle_{*,\omega,q}.$$

Consequently, we see that

$$\langle \Phi z, t\rangle_{*,\omega,q} = \frac{[z,t](b)}{[z,t](\omega_0)} \langle z, \Phi t\rangle_{*,\omega,q}, \tag{9}$$

where

$$[z,t](x) := \left(t(x) \odot D^{*}_{\omega,q} z\left(h^{-1}(x)\right)\right) \ominus \left(z(x) \odot D^{*}_{\omega,q} t\left(h^{-1}(x)\right)\right).$$

By (5) and (6), we conclude that

$$\langle \Phi z, t\rangle_{*,\omega,q} = \langle z, \Phi t\rangle_{*,\omega,q}. \tag{10}$$

□

Theorem 4.3. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proof. Let ξ, η be two distinct eigenvalues with corresponding eigenfunctions z, t, respectively. By (10), we have

$$\langle \Phi z, t\rangle_{*,\omega,q} = \langle z, \Phi t\rangle_{*,\omega,q}$$

$$\langle e^{\xi} \odot z, t\rangle_{*,\omega,q} = \langle z, e^{\eta} \odot t\rangle_{*,\omega,q}$$

$$e^{\xi-\eta}\langle z, t\rangle_{*,\omega,q} = 1.$$

Since $\xi \neq \eta$, we get

$$\langle z, t\rangle_{*,\omega,q} = 1.$$

□

The $(\omega, q)^*$-Wronskian is defined by the formula

$$W_{*,\omega,q}(z,t) = \left(z \odot D^*_{\omega,q}t\right) \ominus \left(t \odot D^*_{\omega,q}z\right).$$

Then we have the following.

Theorem 4.4. *Any two solutions of Eq. (4) are multiplicative linearly dependent if and only if $W_{*,q} = 1$.*

Proof. Let z and t be two multiplicative linearly dependent solutions of Eq. (4), i.e, $z = t^k$, where $k \neq 1$ (see [15]). Then, we have

$$W_{*,\omega,q}(z,t) = \left(z \odot D^*_{\omega,q}t\right) \ominus \left(t \odot D^*_{\omega,q}z\right) = \left(t^k \odot D^*_{\omega,q}t\right) \ominus \left(t \odot D^*_{\omega,q}t^k\right) = 1.$$

Conversely, let $W_{*,\omega,q}(z,t) = \left(z \odot D^*_{\omega,q}t\right) \ominus \left(t \odot D^*_{\omega,q}z\right) = 1$. Then,

$$D^*_{\omega,q}t^{\ln z} = D^*_{\omega,q}z^{\ln t}$$

$$e^{\ln z D_{\omega,q}\ln t} = e^{\ln t D_{\omega,q}\ln z}$$

$$\ln z D_{\omega,q}\ln t - \ln t D_{\omega,q}\ln z = \begin{vmatrix} \ln z & \ln t \\ D_{\omega,q}\ln z & D_{\omega,q}\ln t \end{vmatrix} = 0.$$

Consequently, $\ln z$ and $\ln t$ are linearly dependent (see [4]), i.e., $\ln z = k \ln t$, where $k \neq 1$. □

Theorem 4.5. *The $(\omega,q)^*$-Wronskian of any two solutions of Eq. (4) is independent of x.*

Proof. Let z and t be two solutions of Eq. (4). By (9), we see that

$$\langle \Phi z, t\rangle_{*,\omega,q} = \frac{[z,t](b)}{[z,t](\omega_0)} \langle z, \Phi t\rangle_{*,\omega,q}.$$

Since $\Phi z = e^{\lambda} \odot z$ and $\Phi t = e^{\lambda} \odot t$, we obtain

$$\frac{[z,t](b)}{[z,t](\omega_0)} = 1.$$

Thus,

$$[z,t](b) = [z,t](\omega_0) = W_{*,\omega,q}(z,t)(\omega_0).$$

□

Theorem 4.6. *All eigenvalues of (4)-(6) are simple from the geometric point of view.*

Proof. Let ξ be an eigenvalue with eigenfunctions $z(.)$ and $t(.)$. From (5), we deduce that

$$W_{*,\omega,q}(z,t)(\omega_0) = \left(z(\omega_0) \odot D^*_{\omega,q} t(\omega_0)\right) \ominus \left(t(\omega_0) \odot D^*_{\omega,q} z(\omega_0)\right) = 1,$$

i.e., z and t are multiplicative linearly dependent. □

Now, we shall construct Green's function for the following problem

$$\left(\left(D^*_{-\omega q^{-1},q^{-1}}\right)^{1/q} D^*_{\omega,q} z(x)\right) \oplus \left(e^{r(x)-\lambda} \odot z(x)\right) = e^{g(x)}, \tag{11}$$

where $x \in [\omega_0, b]$, $r(.)$ is a real-valued continuous function at ω_0 defined on $[\omega_0, b]$ and $e^{g(x)} \in L^2_{*,\omega,q}(\omega_0, b)$, which fulfills the supplementary conditions

$$\left(e^{\cos\gamma} \odot z(\omega_0)\right) \oplus \left(e^{\sin\gamma} \odot D^*_{\omega,q} z(\omega_0)\right) = 1, \tag{12}$$

$$\left(e^{\cos\delta} \odot z(a)\right) \oplus \left(e^{\sin\delta} \odot D^*_{\omega,q} z\left(h^{-1}(b)\right)\right) = 1, \tag{13}$$

where $\gamma, \delta \in \mathbb{R}$.

Let $\Xi(x,\lambda)$ and $\Psi(x,\lambda)$ be two basic solutions of Eq. (4) which satisfy the following initial conditions

$$\Xi(\omega_0) = e^{-\sin\gamma},\ D^*_{\omega,q}\Xi(\omega_0) = e^{\cos\gamma},$$
$$\Psi(b) = e^{-\sin\delta},\ D^*_{\omega,q}\Psi\left(h^{-1}(b)\right) = e^{\cos\delta}.$$

We see at once that

$$\omega(\lambda) = -W_{*,\omega,q}(\Xi,\Psi) \neq 1.$$

Theorem 4.7. *If λ is not an eigenvalue of (4)-(6), then the HMSL problem (11)-(13) is solvable for any function $e^{g(x)}$, i.e., the function*

$$z(x,\lambda) = \langle G(x,t,\lambda), e^{g(x)}\rangle_{*,\omega,q}, \tag{14}$$

where

$$G(x,t,\lambda) = e^{-\frac{1}{\omega(\lambda)}} \odot \begin{cases} \Psi(x,\lambda)\odot\Xi(t,\lambda), & \omega_0 \le t \le x \\ \Xi(x,\lambda)\odot\Psi(t,\lambda), & x < t \le b, \end{cases} \tag{15}$$

is the solution of the problem (11)-(13). Conversely, if λ is an eigenvalue of (4)-(6), then the HMSL problem (11)-(13) is generally unsolvable.

Proof. Suppose that λ is not an eigenvalue of (4)-(6). We shall use the method of multiplicative variations of constants. Assume that a particular solution of (11) is given by

$$z(x,\lambda) = \Xi(x,\lambda)^{k_1(x)}\,\Psi(x,\lambda)^{k_2(x)},$$

where $k_1(x)$ and $k_2(x)$ are solutions of the following equations

$$D_{\omega,q}k_1(x) = \frac{qg(h(x))\ln\Psi(h(x))}{\omega(\lambda)},$$

$$D_{\omega,q}k_2(x) = -\frac{qg(h(x))\ln\Xi(h(x))}{\omega(\lambda)}.$$

Hence, we get

$$k_1(x) = k_1(b) - \int_x^b \frac{qg(h(t))\ln\Psi(h(t))}{\omega(\lambda)}d_{\omega,q}t,$$

$$k_2(x) = k_2(\omega_0) - \int_{\omega_0}^x \frac{qg(h(t))\ln\Xi(h(t))}{\omega(\lambda)}d_{\omega,q}t.$$

Then, the general solution of (11) is given by

$$z(x,\lambda)=\Xi(x,\lambda)^{c_1}\Psi(x,\lambda)^{c_2}$$

$$\times\Xi(x,\lambda)^{-\int_x^b\frac{qg(h(t))\ln\Psi(h(t))}{\omega(\lambda)}d_{\omega,q}t}\Psi(x,\lambda)^{-\int_{\omega_0}^x\frac{qg(h(t))\ln\Xi(h(t))}{\omega(\lambda)}d_{\omega,q}t},$$

where $x\in[\omega_0,a]$ and c_1,c_2 are arbitrary constants. By (12) and (13), simple calculations imply

$$c_1=-\int_{\omega_0}^{h^{-1}(b)}\frac{qg(h(t))\ln\Psi(h(t))}{\omega(\lambda)}d_{\omega,q}t$$

$$c_2=-\int_{\omega_0}^{b}\frac{qg(h(t))\ln\Xi(h(t))}{\omega(\lambda)}d_{\omega,q}t.$$

Consequently

$$z(x,\lambda)=\Xi(x,\lambda)^{-\int_{\omega_0}^{h^{-1}(b)}\frac{qg(h(t))\ln\Psi(h(t))}{\omega(\lambda)}d_{\omega,q}t-\int_x^b\frac{qg(h(t))\ln\Psi(h(t))}{\omega(\lambda)}d_{\omega,q}t}$$

$$\times\Psi(x,\lambda)^{-\int_{\omega_0}^x\frac{qg(h(t))\ln\Xi(h(t))}{\omega(\lambda)}d_{\omega,q}t-\int_{\omega_0}^b\frac{qg(h(t))\ln\Xi(h(t))}{\omega(\lambda)}d_{\omega,q}t}$$

$$=\Xi(x,\lambda)^{-\int_{\omega_0}^x\frac{g(t)\ln\Psi(t)}{\omega(\lambda)}d_{\omega,q}t}\Psi(x,\lambda)^{-\int_x^b\frac{g(t)\ln\Xi(t)}{\omega(\lambda)}d_{\omega,q}t},$$

and the proof is complete. In fact, by (14) we have

$$\begin{aligned}z(x,\lambda)&=\langle G(x,t,\lambda),e^{g(x)}\rangle_{*,q}\\&=e^{\int_{\omega_0}^x g(t)\ln G(x,t,\lambda)d_{\omega,q}t}.e^{\int_x^b g(t)\ln G(x,t,\lambda)d_{\omega,q}t}.\end{aligned}\tag{16}$$

From (15), we see that

$$G(x,t,\lambda)=\begin{cases}\left(\Psi(x,\lambda)^{\ln\Xi(t,\lambda)}\right)^{-\frac{1}{\omega(\lambda)}}, & \omega_0\le t\le x\\ \left(\Xi(t,\lambda)^{\ln\Psi(x,\lambda)}\right)^{-\frac{1}{\omega(\lambda)}}, & x<t\le b.\end{cases}\tag{17}$$

We conclude from (16) and (17) that

$$z(x,\lambda)=\Xi(x,\lambda)^{-\int_{\omega_0}^x\frac{g(t)\ln\Psi(t)}{\omega(\lambda)}d_{\omega,q}t}\Psi(x,\lambda)^{-\int_x^b\frac{g(t)\ln\Xi(t)}{\omega(\lambda)}d_{\omega,q}t}.$$

□

Theorem 4.8. *Green's function $G(x,t,\lambda)$ defined by (17) is unique.*

Proof. Assume that there is another Green's function $\widetilde{G}(x,t,\lambda)$ for the problem (11)-(13). Then, we get

$$z(x,\lambda) = \langle \widetilde{G}(x,t,\lambda), e^{g(x)} \rangle_{*,q}.$$

Thus

$$\langle G(x,t,\lambda) \ominus \widetilde{G}(x,t,\lambda), e^{g(x)} \rangle_{*,q} = 0. \tag{18}$$

Putting $g(x) = \ln\left[G(x,t,\lambda) \ominus \widetilde{G}(x,t,\lambda)\right]$ in (18), we obtain

$$G(x,t,\lambda) = \widetilde{G}(x,t,\lambda).$$

□

Theorem 4.9. *Green's function $G(x,t,\lambda)$ defined by (17) satisfies the following properties.*
i) $G(x,t,\lambda)$ is continuous at (ω_0,ω_0).
ii) $G(x,t,\lambda) = G(t,x,\lambda)$.
iii) For each fixed $t \in (\omega_0, b]$, as a function of x, $G(x,t,\lambda)$ satisfies Eq. (11) in the intervals $[\omega_0,t),(t,b]$ and it satisfies (12)-(13).

Proof. *i*) Since $\Psi(.,\lambda)$ and $\Xi(.,\lambda)$ are continuous at ω_0, it may be concluded that $G(x,t,\lambda)$ is continuous at (ω_0,ω_0).
ii) Easy to be checked.
iii) Let $t \in (\omega_0, b]$ be fixed and $x \in [\omega_0, t]$. Then, we see that

$$G(x,t,\lambda) = \Psi(x,\lambda)^{\frac{-\ln \Xi(t,\lambda)}{\omega(\lambda)}}.$$

It follows that

$$\Phi G(x,t,\lambda) = e^{\lambda} \odot G(x,t,\lambda).$$

Similarly for $x \in (t,b]$.

$$\begin{aligned}
&\left(e^{\cos\gamma} \odot G(\omega_0,t,\lambda)\right) \oplus \left(e^{\sin\gamma} \odot D^*_{\omega,q} G(\omega_0,t,\lambda)\right) \\
&= \left[\Xi(\omega_0)^{\cos\gamma} D^*_{\omega,q} \Xi(\omega_0)^{\sin\gamma}\right]^{\frac{-\ln \Psi(t)}{\omega(\lambda)}} = 1, \\
&\left(e^{\cos\delta} \odot G(b,t,\lambda)\right) \oplus \left(e^{\sin\delta} \odot D^*_{\omega,q} G\left(h^{-1}(b),t,\lambda\right)\right) \\
&= \left[\Psi(b)^{\cos\delta} D^*_{\omega,q} \Xi\left(h^{-1}(b)\right)^{\sin\delta}\right]^{\frac{-\ln \Xi(t)}{\omega(\lambda)}} = 1.
\end{aligned}$$

□

Conclusion. In this study, we introduce the Hahn multiplicative calculus. Basic theorems are proved within this calculus. Next the Hahn multiplicative Sturm–Liouville is defined. Some spectral properties of this problem were studied. Finally, Green's function is created for this problem. Some properties of Green's function have been given.

REFERENCES

[1] K. A. Aldwoah, *Generalized time scales and associated difference equations.* Ph.D. Thesis, Cairo University (2009).

[2] B.P. Allahverdiev - H. Tuna, *q-multiplicative Dirac system,* Konuralp J. Math., (In press).

[3] M. H. Annaby - A. E. Hamza -K. A. Aldwoah, *Hahn difference operator and associated Jackson–Nörlund integrals.* J. Optim. Theory Appl. 154 (2012), 133-153.

[4] M. H. Annaby - A. E. Hamza - S. D. Makharesh, *A Sturm–Liouville theory for Hahn difference operator.* in: Xin Li, Zuhair Nashed (Eds.), Frontiers of Orthogonal Polynomials and *q*-Series, World Scientific, Singapore, (2018), 35-84.

[5] A. E. Bashirov -E. M. Kurpinar -A. *Ozyapici, Multiplicative calculus and its applications.* J. Math. Anal. Appl. 337 (1) (2008), 36-48.

[6] S. Goktas - E. Yilmaz - A. C. Yar, Multiplicative derivative and its basic properties on time scales. Math. Meth. Appl. Sci. 45 (2022), 2097-2109.

[7] S. Goktas - H. Kemaloglu - E. Yilmaz, *Multiplicative conformable fractional Dirac system.* Turkish J. Math. 46 (2022), 973-990.

[8] S. Goktas, *A New Type of Sturm–Liouville equation in the non-Newtonian calculus.* J. Funct. Spaces. 5203939 (2021), 1-8.

[9] M. Grossman, *An introduction to Non-Newtonian calculus.* Intern. J. Math. Educ. Sci. Techn. 10 (4) (1979), 525-528.

[10] M. Grossman - R. Katz, *Non-Newtonian calculus.* Pigeon Cove, MA: Lee Press, 1972.

[11] T. Gulsen - E. Yilmaz - S. Goktas, *Multiplicative Dirac system.* Kuwait J. Sci. 2022. doi:10.48129/kjs.13411.

[12] W. Hahn, *Über orthogonalpolynome, die q-Differenzengleichungen genügen.* Math. Nachr. 2 (1949), 4–34.

[13] F. H. Jackson, *q-Difference equations,* Amer. J. Math., 32 (1910), 305-314.

[14] D. L. Jagerman, *Difference Equations with Applications to Queues.* Dekker, New York, 2000.

[15] N. Yalcin - E. Celik, *Solution of multiplicative homogeneous linear differential*

equations with constant exponentials, New Trends Math. Sci. 6 (2) (2018), 58-67.

[16] G. Yener - I. Emiroglu, *A q-analogue of the multiplicative calculus : q-multiplicative calculus*, Discr. Contin. Dynam. Syst.-Ser. S, 8 (6) (2015), 1435-1450.

B. P. ALLAHVERDIEV
Department of Mathematics, Khazar University,
AZ1096 Baku, Azerbaijan e-mail: bilenderpasaoglu@gmail.com

H. TUNA
Department of Mathematics, Burdur Mehmet Akif Ersoy University,
15030 Burdur, Turkey e-mail: hustuna@gmail.com