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# ON OSCILLATION PROPERTIES OF THE EIGENFUNCTIONS OF A FOURTH ORDER DIFFERENTIAL OPERATOR

**Abstract**

*The spectral problem for a fourth order ordinary differential operator is investigated. The oscillation properties of the eigenfunctions and their derivatives are established.*

Let's consider the boundary-value problem

$$\left(p\left(x\right)y''\right)'' - \left(q\left(x\right)y'\right)' = \lambda\rho\left(x\right)y, \quad 0 < x < l, \tag{1}$$

$$y'\left(0\right)\cos\alpha - \left(py''\right)\left(0\right)\sin\alpha = 0, \tag{2.a}$$

$$y\left(0\right)\cos\beta + Ty\left(0\right)\sin\beta = 0, \tag{2.b}$$

$$y'\left(l\right)\cos\gamma + \left(py''\right)\left(l\right)\sin\gamma = 0, \tag{2.c}$$

$$y\left(l\right)\cos\delta - Ty\left(l\right)\sin\delta = 0, \tag{2.d}$$

where $\lambda$ is a spectral parameter, the functions $p\left(x\right), q\left(x\right), \rho\left(x\right)$ are strictly positive and continuous on $\left[0, l\right], p\left(x\right)$ has absolutely continuous derivative, $q\left(x\right)$ is absolutely continuous on $\left[0, l\right]$ and $\alpha, \beta, \gamma, \delta$ are real constants, such that $0 \leq \alpha, \beta, \gamma \leq \pi/2, \pi/2 < \delta < \pi$ and

$$Ty = \left(py''\right)' - qu'. \tag{3}$$

The present paper is devoted to study of oscillation properties of the eigenfunctions of oscillation properties of the eigenfunctions of boundary-value problem (1)-(2). The basic result of this paper is the oscillation theorem (theorem 4).

The oscillation properties of the eigenfunctions of boundary-value problem (1)-(2) provided $0 \leq \delta \leq \pi/2$ have been investigated in detail in [1]. In this work it is investigated only positive eigenvalues and corresponding eigenfunctions of problem (1)-(2). In this connection in the paper [1] the following two cases are excluded: (i) $\alpha = \gamma = 0$ and $\beta = \delta = \pi/2$, (ii) any three of parameters $\alpha, \beta, \gamma, \delta$ are equal to $\pi/2$. In reality, only the case $\beta = \delta = \pi/2$ is to be excluded. Let's prove this.

It is known, that the least eigenvalue of boundary-value problem (1)-(2) is a minimum of Relay's ration

$$R\left[y\right] = \left(\int\limits_0^l \left(py''^2 + qy'^2\right)dx + N\left[y\right]\right)\left(\int\limits_0^l \rho y^2 dx\right)^{-1}, \tag{4}$$

where $N\left[y\right]$ is a functional, which takes only nonnegative values (see [2, p.160) or [1, p.64]).

Let $\beta = \delta = \pi/2$. The direct testing shows, that the function $y(x) \equiv c_0 = const \neq 0$ $(x \in [0,l])$ is an eigenfunction of boundary-value problem (1)-(2), corresponding to the eigenvalue $\lambda = 0$. The simplicity of the eigenvalue $\lambda = 0$ follows from the fact, that the corresponding eigenfunction $y(x)$ must satisfy the relation $y'(x) \equiv 0$ $(x \in [0,l])$ (see (4)).

Let $\lambda = 0$ is an eigenvalue of boundary-value problem (1)-(2). From formula (4) it follows, that for the corresponding eigenfunction $y(x)$ it is true $y'(x) \equiv 0$ $(x \in [0,l])$, that is equivalent to $y(x) \equiv c_0 = const \neq 0$ $(x \in [0,l])$. Boundary conditions (2a) and (2c) are automatically satisfied at that. For fulfillment of boundary conditions (2b) and (2d) the condition $\beta = \delta = \pi/2$ is to be fulfilled.

As in [1], to study the oscillation properties of eigenfunctions and their derivatives we'll use the Prufer-type transformation

$$u(x) = r(u) \sin\psi(x) \cos\theta(x), \qquad (5.a)$$

$$u'(x) = r(x) \cos\psi(x) \sin\varphi(x), \qquad (5.b)$$

$$\left(pu''\right)(x) = r(x) \cos\psi(x) \cos\varphi(x), \qquad (5.c)$$

$$Tu(x) = r(x) \sin\psi(x) \sin\theta(x). \qquad (5.d)$$

Let's write equation (1) in equivalent form

$$U' = MU, \qquad (6)$$

where

$$U = \begin{pmatrix} y \\ y' \\ py'' \\ Ty \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/p & 0 \\ 0 & q & 0 & 1 \\ \lambda\rho & 0 & 0 & 0 \end{pmatrix}.$$

Assuming $w(x) = ctg\psi(x)$ and using transformation (5) in (6) we'll obtain the system of first order differential equations with respect to the functions $r, w, \theta, \varphi$ of the following form:

$$r' = [\sin 2\psi\theta \sin\varphi + \left(q + \frac{1}{p}\right)\cos^2\psi \sin 2\varphi +$$

$$+ \sin 2\psi \sin\theta \cos\varphi + \frac{\lambda\rho}{2}\sin^2\psi \sin 2\theta]\frac{r}{2}, \qquad (7.a)$$

$$w' = -w^2 \cos\theta \sin\varphi + \frac{1}{2}\left(q + \frac{1}{p}\right) w \sin 2\varphi + \sin\theta\cos\varphi - \frac{\lambda\rho}{2} w \sin 2\theta, \qquad (7.b)$$

$$\theta' = -w \sin\varphi \sin\theta + \lambda\rho\cos^2\theta, \qquad (7.c)$$

$$\varphi' = \frac{1}{p}\cos^2\varphi - q\sin^2\varphi - \frac{1}{w}\sin\theta\sin\varphi. \qquad (7.d)$$

Let's cite some statements from [1].

**Lemma 1. (see [1], p.59, lemma 2.1).** *Let $y(x,\lambda)$ be a nontrivial solution of differential equation (1) at $\lambda>0$. If $y,y',y''$ and $Ty$ are nonnegative at $x=a$ (but not all zero), then they all are positive for $x>a$. If $y,-y',y''$* and *$-Ty$ are nonnegative at $x=a$ (but not all zero), then they all are positive for $x<a$.*

**Theorem 1. (see [1], p.61, theorem 3.1).** *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.c) at $\lambda>0$. Then the Jacobian $J[y]=r^3\cos\psi$ $\sin\psi$ of the transformation (5) does not vanish in $(0,l)$.*

The following lemma holds:

**Lemma 2.** *At every fixed $\lambda\in\mathbf{C}$ there exits the unique (to within constant factor) nontrivial solution $y(x,\lambda)$ of problem (1), (2.a), (2.b), (2.c).*

**Proof.** Denote by $\varphi_k(x,\lambda)$ $\left(k=\overline{1,4}\right)$ the solutions of equation (1), normalized at $x=0$ by Cauchy conditions

$$\varphi_k^{(s-1)}(0,\lambda)=\delta_{ks}\quad\left(s=\overline{1,3}\right),\quad T\varphi_k(0,\lambda)=\delta_{k4},\tag{8}$$

where $\delta_{ks}$ is a Kronecker's symbol.

We'll search the function $y(x,\lambda)$ in the form

$$y(x,\lambda)=\sum_{k=1}^{4}C_k\varphi_k(x,\lambda),\tag{9}$$

where $C_k$ $\left(k=\overline{1,4}\right)$ are some constants.

Suppose, that in boundary conditions (2.a), (2.b), (2,c) $\alpha\neq0$, $\beta\neq0$, $\gamma\neq0$. From (8), (9) and from boundary conditions (2.a), (2.b) it follows, that

$$C_3=\frac{C_2}{p(0)}ctg\alpha,\quad C_4=-C_1ctg\beta$$

holds. From here and from (9) we'll obtain

$$y(x,\lambda)=C_1\left\{\varphi_1(x,\lambda)-\varphi_4(x,\lambda)\,ctg\beta\right\}+C_2\left\{\varphi_2(x,\lambda)+\varphi_3(x,\lambda)\frac{ctg\alpha}{p(0)}\right\}.\tag{10}$$

Taking into account (8), (10) and (2.c) for definition of $C_1$ and $C_2$ we'll obtain the relation

$$C_1\alpha^*(\lambda)+C_2\beta^*(\lambda)=0,$$

where

$$\alpha^*(\lambda)=\left\{\varphi_1'(l,\lambda)\,ctg\gamma+p(l)\,\varphi_1''(l.\lambda)\right\}-ctg\beta\left\{\varphi_4'(l.\lambda)\,ctg\gamma+p(l)\,\varphi_4''(l,\lambda)\right\},\tag{11}$$

$$\beta^*(\lambda)=\left\{\varphi_2'(l,\lambda)\,ctg\gamma+p(l)\,\varphi_2''(l.\lambda)\right\}-\frac{ctg\alpha}{p(0)}\left\{\varphi_3'(l.\lambda)\,ctg\gamma+p(l)\,\varphi_3''(l,\lambda)\right\}.\tag{12}$$

To complete the proof of lemma 2 in the considered case it suffices to show, that it holds

$$|\alpha^*(\lambda)|+|\beta^*(\lambda)|>0.\tag{13}$$

From lemma 1 and from (8) it follows, that at $\lambda > 0$ the inequalities $\varphi'_k(l,\lambda) > 0$, $\varphi''_k(l,\lambda) > 0$ $\left(k = \overline{1,4}\right)$ are true. From here and from (12) obtain the truth of (13) at $\lambda > 0$.

Let $\lambda \in \mathbf{C}/\mathbf{R}^+$. Let's prove the truth of (13). Really, otherwise the functions

$$\phi_1(x,\lambda) = \varphi_1(x,\lambda) - ctg\beta\varphi_4(x,\lambda), \ \phi_2(x,\lambda) = \varphi_2(x,\lambda) + \frac{ctg\alpha}{p(0)}\varphi_3(x,\lambda) \tag{14}$$

are the solutions of problem (1), (2.a), (2.b), (2.c). It is obvious, that any linear combination of the functions $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ is also the solution of this problem. The eigenvalues of boundary-value problem (1)-(2) at $\delta = 0$ are positive (see [1] or theorem 3 of the given paper). Hence, $\phi_1(l,\lambda) \neq 0$ and $\phi_2(l,\lambda) \neq 0$. Let's define the function $\upsilon(x,\lambda)$ by the following way:

$$\upsilon(x,\lambda) = \phi_1(x,\lambda)\phi_2(l,\lambda) - \phi_2(x,\lambda)\phi_1(l,\lambda).$$

It is obvious, that $\upsilon(l,\lambda) = 0$. Then the function $\upsilon(x,\lambda)$ is an eigenfunction of problem (1)-(2) at $\delta = 0$, corresponding to the eigenvalue $\lambda \in \mathbf{C}/\mathbf{R}^+$. The obtained contradiction proves the truth of (13).

The rest cases are considered similarly. Lemma 2 is proved.

**Remark 1.** *From proof of lemma 2 it is obvious, that solution of problem (1), (2.a), (2.b), (2.c), i.e. the function $y(x,\lambda)$ for each fixed $x \in [0,l]$ may be considered an entire function of $\lambda$. In particular, in the case $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, the function $y(x,\lambda)$ has the form*

$$y(x,\lambda) = \beta^*(\lambda)\phi_1(x,\lambda) - \alpha^*(\lambda)\phi_2(x,\lambda),$$

*where $\alpha^*(\lambda), \beta^*(\lambda), \phi_1(x,\lambda), \phi_2(x,\lambda)$ are defined by relations (11), (12) and (14). As the functions $\varphi_k(x,\lambda)$ $\left(k = \overline{1,4}\right)$ and their derivatives for each fixed $x \in [0,l]$ are entire functions of $\lambda$, then $y(x,\lambda)$ for each fixed $x \in [0,l]$ is also an entire function of $\lambda$.*

**Lemma 3.** *The eigenvalues of boundary-value problem (1)-(2) are real and form no more than countable set, having no finite limit points. All eigenvalues of boundary-value problem (1)-(2) are simple.*

**Proof.** The reality of eigenvalues follows from self-adjointness of boundary-value problem (1)-(2).

Let $y(x,\lambda)$ be a solution of problem (1), (2.a), (2.b), (2.c). Then the eigenvalues of problem (1)-(2) are the roots of the equation

$$\Phi(\lambda) \equiv y(l,\lambda)\cos\delta - Ty(l,\lambda)\sin\delta = 0. \tag{15}$$

The entire function $\Phi(\lambda)$ doesn't vanish at nonreal $\lambda$. Consequently, it is not equal to zero identically. Therefore, its zeros form no more than countable set, having no finite limit point.

By virtue of (1) we have

$$(Ty(x,\mu))'\,y(x,\lambda) - (Ty(x,\lambda))'\,y(x,\mu) = (\mu - \lambda)\rho(x)\,y(x,\lambda)\,y(x,\mu).$$

Integrating this identity in limits from 0 to $l$, using the formula of integration by parts and taking into account (2.a), (2.b), (2.c) we obtain

$$y(l,\lambda)\,Ty(l,\mu)-y(l,\mu)\,Ty(l,\lambda)=(\mu-\lambda)\int_0^l\rho(x)\,y(x,\lambda)\,y(x,\mu)\,dx. \tag{16}$$

Deriving the both parts of (16) by $(\mu-\lambda)$ and by the next limiting passage as $\mu\to\lambda$ we'll obtain

$$y(l,\lambda)\frac{\partial}{\partial\lambda}Ty(l,\lambda)-Ty(l,\lambda)\frac{\partial}{\partial\lambda}y(l,\lambda)=\int_0^l\rho(x)\,y^2(x,\lambda)\,dx\ . \tag{17}$$

Let's prove, that equation (15) has only simple roots. Really, if $\lambda=\lambda^*$ is a multiple root of equation (15), then the equalities

$$y(l,\lambda^*)\cos\delta-Ty(l,\lambda^*)\sin\delta=0,$$

$$\cos\delta\frac{\partial}{\partial\lambda}y(l,\lambda^*)-\sin\delta\frac{\partial}{\partial\lambda}Ty(l,\lambda^*)=0$$

hold.

Using the last two equalities in (17) at $\lambda=\lambda^*$ we have $\int_0^l\rho(x)\,y^2(x,\lambda^*)\,dx=0$, that is contradiction. Lemma 3 is proved.

**Lemma 4.** *Let $y(x,\lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) and one of the following conditions be fulfilled: (i) $\lambda<0$; (ii) $\lambda=0$, $\beta\in[0,\pi/2)$. Then Jacobian $J[y]=r^3\cos\psi\sin\psi$ of the transformation (5) does not vanish in $(0,l)$.*

**Proof.** Suppose, that the statement of lemma 4 is not true and at the some point $x_1\in(0,l)$ it holds $\sin\psi\cos\psi=0$. The following cases are possible: (a) $\sin\psi(x_1,\lambda)=0$; (b) $\cos\psi(x_1,\lambda)=0$.

Let $\lambda<0$. Let's consider case (a). Then by virtue of (5) it holds $y(x_1,\lambda)=Ty(x_1,\lambda)=0$. Suppose, that $y(x,\lambda)>0$ at the left neighbourhood $U(x_1)$ of the point $x_1$. Then from (1) it follows, that $(Ty(x,\lambda))'<0$ at $x\in U(x_1)$. So, $Ty(x,\lambda)>0$ at $x\in U(x_1)$. From (2.b) it follows, that $y(0,\lambda)\,Ty(0,\lambda)\le 0$. Then there exists the point $x_0\in[0,x_1)$ such that $y(x_0,\lambda)\,Ty(x_0,\lambda)=0$ and

$$y(x,\lambda)\,Ty(x,\lambda)>0\qquad(x_0<x<x_1)\,. \tag{18}$$

Let $Ty(x_0,\lambda)=0$. Hence, there exists the point $\xi_0\in(x_0,x_1)$ such that $(Ty(x,\lambda))'_{x=\xi_0}=0$. From here and from equation (1) we obtain $y(\xi_0,\lambda)=0$. The last equality contradicts to inequality (18).

Let $y(x_0,\lambda)=0$. Hence, there exists the point $\eta_0\in(x_0,x_1)$ such that $y'(\eta_0,\lambda)=0$. It is obvious, that $y(\eta_0,\lambda)>0$, $\quad Ty(\eta_0,\lambda)>0$. Let's define the number $\delta_0\in\left(0,\frac{\pi}{2}\right)$ by the following way: $\delta_0=arctg\frac{y(\eta_0,\lambda)}{Ty(\eta_0,\lambda)}$. So, the function $y(x,\lambda)$ is a solution of boundary-value problem (1)-(2) at $l=\eta_0$, $\gamma=0$, $\delta=\delta_0$. As the eigenvalues of boundary-value problem (1)-(2) at $l=\eta_0$, $\gamma=0$, $\delta=\delta_0$ are positive, then we obtain the contradiction. Hence, $\sin\psi(x,\lambda)\neq 0\quad(0<x<l)$ at $\lambda<0$.

Let $\lambda < 0$ and (b) hold. By virtue of (5) we have $y'(x_1,\lambda) = y''(x_1,\lambda) = 0$. It is obvious, that $y(x_1,\lambda) \neq 0$ and $Ty(x_1,\lambda) \neq 0$. Really, if $y(x_1,\lambda) = 0$, then $y(x,\lambda)$ is an eigenfunctions of boundary-value problem (1)-(2) at $\gamma = \pi/2$, $\delta = 0$, $l = x_1$, that contradicts to the condition $\lambda < 0$. By the similar way the case $Ty(x_1,\lambda) = 0$ is excluded.

As $Ty(x_1,\lambda) \neq 0$, then it is obvious, that the point $x_1$ is a point of local extremum of the function $y'(x,\lambda)$. Suppose, that $y'(x,\lambda) > 0$ at the deleted neighbourhood $V(x_1)$ of the point $x_1$. Then $y''(x,\lambda) < 0$ at the left neighbourhood $V^-(x_1)$ of the point $x_1$ and $y''(x,\lambda) > 0$ at the right neighbourhood $V^+(x_1)$ of the point $x_1$. From here and from condition (2.a) it follows, that there exists the point $x_0 \in [0,x_1)$ such that $y'(x_0,\lambda)\, y''(x_0,\lambda) = 0$ and

$$y'(x,\lambda) > 0, \quad y''(x,\lambda) < 0 \qquad (x \in (x_0,x_1)). \tag{19}$$

Suppose, that $y'(x_0,\lambda) = 0$. Then there exists the point $\xi_0 \in (x_0,x_1)$ such that $y''(\xi_0,\lambda) = 0$. The last relation contradicts to (19).

Let $y''(x_0,\lambda) = 0$. Then there exists the point $\xi_0 \in (x_0,x_1)$ such that $(p(x)\, y''(x,\lambda))'_{x=\xi_0} = 0$. From (19) it follows, that

$$Ty(x_0,\lambda) = \big(p(x)\, y''(x,\lambda)\big)'_{x=\xi_0} - q(\xi_0)\, y'(\xi_0,\lambda) < 0.$$

Besides, $Ty(x_1,\lambda) = (p(x)\, y''(x,\lambda))'_{x=x_1} - q(x_1)\, y'(x_1,\lambda) = p(x_1)\, y'''(x_1,\lambda) > 0$. Hence, there exists the point $\eta_0 \in (\xi_0,x_1)$ such that $Ty(\eta_0,\lambda) = 0$.

We'll define the number $\gamma_0 \in \left(0,\dfrac{\pi}{2}\right)$ by the following equality:

$$\gamma_0 = -arctg\frac{p(\eta_0)\, y''(\eta_0,\lambda)}{y'(\eta_0,\lambda)}.$$

It is easy to check, that $y(x,\lambda)$ is an eigenfunction of boundary-value problem (1)-(2) at $\gamma = \gamma_0$, $\delta = \pi/2$, $l = \eta_0$, that contradicts to the condition $\lambda < 0$.

Let now $\lambda = 0$, $\beta \in [0,\pi/2)$. Let's consider case (a). From (1) it follows, that $Ty(x,0) \equiv const$ $(0 \le x \le l)$. Hence, by virtue (5.d) we have: $Ty(x,0) \equiv 0$ $(0 \le x \le l)$. Multiplying this equality by the function $y(x,\lambda)$ and integrating the obtained identity from 0 to $l$, we obtain

$$p(l)\, y''(l,0)\, y'(l,0) - p(0)\, y''(0,0)\, y'(0,0) -$$
$$- \int_0^l \big(p(x)\, y''^2(x,0) + q y'^2(x,0)\big)\, dx = 0. \tag{20}$$

By virtue of conditions (2.a) and (2.c) we have

$$p(l)\, y''(l,0)\, y'(l,0) \le 0, \quad p(0)\, y''(0,0)\, y'(0,0) \ge 0. \tag{21}$$

From here and from (20) we obtain, that $y(x,0) \equiv const$. As $\sin\psi(x_1,0) = 0$, then $y(x,0) \equiv 0$ $(0 \le x \le l)$, that is contradiction.

Let $\lambda = 0$, $\beta \in [0, \pi/2)$ and $\cos\psi(x_1, 0) = 0$, where $x_1$ is some point from $(0, l)$. By virtue of (5) we have

$$y'(x_1, 0) = y''(x_1, 0) = 0. \tag{22}$$

Let's prove, that in the considered case $Ty(0,0) \neq 0$. Really, if $Ty(0,0) = 0$, then from (2.b) it follows, that $y(0,0) = 0$. Besides, from (1) obtain, that $Ty(x,0) \equiv const = 0$ $(0 \le x \le l)$. Using (20), (21) and taking into account the equality $y(0,0) = 0$, we conclude, that $y(x,0) \equiv 0$ $(0 \le x \le l)$. The last is contradiction.

As $Ty(x,0) = Ty(0,0) \neq 0$ $(0 \le x \le l)$, then from (3) it follows, that $y'''(x_1, 0) \neq 0$. So, $x_1$ is a double zero of the function $y'(x, \lambda)$. Without losing generality, it is possible to consider, that $y'''(x_1, 0) > 0$. Hence, $Ty(x_1, 0) = p(x_1)\, y'''(x_1, 0) > 0$ and besides, at the some right neighbourhood of the point $x_1$ it holds

$$y'(x, 0) > 0, \quad y''(x, 0) > 0. \tag{23}$$

Let's assume, that $(x_1, l_0)$ is an interval of maximum length, where inequality (23) is true. It is obvious, that $y'(l_0, 0) \ge 0$, $\;y''(l_0, 0) \ge 0$.

Let $y'(l_0, 0) = 0$. Then from (22) it follows, that for some point $\xi \in (x_1, l_0)$ it holds $y''(\xi, 0) = 0$. The last contradicts to (23).

Let $y''(l_0, 0) = 0$. As $p(x_1)\, y''(x_1, 0) = p(l_0)\, y''(l_0, 0) = 0$, then again there exists the point $\xi \in (x_1, l_0)$ such that $(p(x)\, y''(x, 0))'_{x=\xi} = 0$. Hence $Ty(\xi, 0) = (p(x)\, y''(x, 0))'_{x=\xi} - q(\xi)\, y'(\xi, 0) < 0$. On the other hand it holds $Ty(x, 0) \equiv const = Ty(x_1, 0)$ $\;(0 \le x \le l)$, that is contradiction.

So, we've shown, that $l_0 = l$ and $y'(l, 0) > 0$, $y''(l, 0) > 0$. The last contradicts to condition (2.c). The proof of lemma 4 is completed.

Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) and either $\lambda \in \mathbf{R}/\{0\}$, or $\lambda = 0$ and $\beta \in [0, \pi/2)$. Suppose, that $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ are corresponding functions from (5). Without losing generality, we can define the initial value of these functions by the following way:

$$\theta(0, \lambda) = \beta - \frac{\pi}{2}, \tag{24}$$

$$\varphi(0, \lambda) = \alpha. \tag{25}$$

The proof of this fact is completely made by scheme of the proof of theorem 3.1 from [3] (see theorem 3.3 from [1]).

The following two statements are proved in [1].

**Theorem 2. (see theorem 4.2 from [1]).** *Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.b), (2.c) at $\lambda > 0$. Then $\theta(l, \lambda)$ is a strictly increasing continuous function of $\lambda$.*

**Theorem 3. (see theorems 5.4 and 5.5 from [1]).** *The eigenvalues of boundary-value problem (1)-(2) at $\delta \in [0, \pi/2]$ (except the case $\beta = \delta = \pi/2$) form infinitely increasing sequence $\{\mu_k(\delta)\}_1^\infty$ such that*

$$0 < \mu_1(\delta) < \mu_2(\delta) < ... < \mu_n(\delta) < ...,$$

$$\theta\left(1, \mu_n(\delta)\right) = (2n-1)\frac{\pi}{2} - \delta. \tag{26}$$

*Besides, the eigenfunction* $\vartheta_n^\delta(x)$, *corresponding to the eigenvalue* $\mu_n(\delta)$, *has exactly* $(n-1)$ *simple zeros in the interval* $(0,l)$, *and the function* $T\vartheta_n^\delta(x)$ *has exactly* $n$ *zeros on the segment* $[0,l]$.

**Remark 2.** *In the case* $\beta = \delta = \pi/2$ *the first eigenvalue of boundary problem (1)-(2) is equal to zero and the corresponding eigenfunction is constant. In this case the statement of theorem 3 is true at* $n \geq 2$.

Obviously, the eigenvalues $\mu_n = \mu_n(0)$ and $\nu_n = \mu_n\left(\frac{\pi}{2}\right)$ $(n \in \mathbf{N})$ are zeros of the entire functions $y(l,\lambda)$ and $Ty(l,\lambda)$, respectively. Besides we note that by theorem 2 and equality (23) the relation $\nu_1 < \mu_1 < \nu_2 < \mu_2 < ...$ is valid.

Let's consider the function $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ at $\lambda \in K \equiv \bigcup\limits_{k=0}^{\infty}\left(\mu_k, \mu_{k+1}\right)$, where $\mu_0 = -\infty$. From (16) at $\lambda, \mu \in K$ we have

$$\frac{Ty(l,\mu)}{y(l,\mu)} - \frac{Ty(l,\lambda)}{y(l,\lambda)} = (\mu-\lambda)\frac{\int\limits_0^l \rho(x)\, y(x,\mu)\, y(x,\lambda)\, dx}{y(l,\mu)\, y(l,\lambda)}. \tag{27}$$

Deriving both parts of (27) by $(\mu-\lambda)$ and by the next limiting passage as $\mu \to \lambda$ we'll obtain

$$\frac{\partial}{\partial\lambda}\left(\frac{Ty(l,\lambda)}{y(l,\lambda)}\right) = \frac{\int\limits_0^l \rho(x)\, y^2(x,\lambda)\, dx}{y^2(l,\lambda)} > 0. \tag{28}$$

So, we proved the following statement.

**Lemma 5.** *The function* $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ *in each of the interval* $\left(\mu_k, \mu_{k+1}\right)$ $(k = 0,1,2,...)$ *is a strictly increasing function of* $\lambda$.

**Lemma 6.** *Let* $y(x,\lambda)$ *be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). Then it holds the relation*

$$\lim_{\lambda\to-\infty}\frac{Ty(l,\lambda)}{y(l,\lambda)} = -\infty. \tag{29}$$

**Proof.** Without losing generality, it may be considered that $\int\limits_0^l \rho(x)\, y^2(x,\lambda)\, dx = 1$. As it is proved in [4, p.353-354] it holds the inequality

$$y^2(l,\lambda) \leq c_0\sqrt{\int\limits_0^1 q(x)\, y'^2(x,\lambda)\, dx} + c_1, \tag{30}$$

where $c_0$ and $c_1$ are positive constants, dependent only on the functions $q(x)$ and $\rho(x)$.

Multiplying both parts of (1) by the function $y(x,\lambda)$ and integrating this identity by $x$ in the limits from 0 to $l$, we'll obtain

$$y(l,\lambda)\, Ty(l,\lambda) - y(0,\lambda)\, Ty(0,\lambda) - p(l)\, y'(l,\lambda)\, y''(l,\lambda) +$$

$$+p(0)\,y'(0,\lambda)\,y''(0,\lambda)+\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx+\int\limits_0^l \rho(x)\,y''^2(x,\lambda)\,dx=\lambda. \tag{31}$$

By virtue of boundary conditions (2.a), (2.b), (2.c) the inequalities

$$p(l)\,y'(l,\lambda)\,y''(l,\lambda)\le 0,\quad y(0,\lambda)\,Ty(0,\lambda)\le 0,\quad p(0)\,y'(0,\lambda)\,y''(0,\lambda)\ge 0$$

are true. From here and from (31) it follows, that

$$\lim_{\lambda\to-\infty} y(l,\lambda)\,Ty(l,\lambda)=-\infty. \tag{32}$$

From lemma 5 it implies, that as $\lambda\to-\infty$, the ratio $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}$ has finite or infinite limit. Suppose, that

$$\lim_{\lambda\to-\infty}\frac{Ty(l,\lambda)}{y(l,\lambda)}=-a_0, \tag{33}$$

where $0<a_0<+\infty$. Taking into account (32) and (33) we'll obtain, that $\lim\limits_{\lambda\to-\infty} y^2(l,\lambda)=+\infty$. From here and from (30) we have

$$\lim_{\lambda\to-\infty}\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx=+\infty. \tag{34}$$

By virtue of (33) at the sufficiently large by module negative values of $\lambda$ the inequality $\left|\dfrac{Ty(l,\lambda)}{y(l,\lambda)}\right|\le a_0$ is true. From here and from (31), (30) at those values of $\lambda$ we'll obtain

$$\lambda\ge\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx-|y(l,\lambda)\,Ty(l,\lambda)|\ge\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx-a_0y^2(l,\lambda)\ge$$

$$\ge\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx-a_0c_0\sqrt{\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx}-a_0c_1\ge$$

$$\ge\sqrt{\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx}\left(\sqrt{\int\limits_0^l q(x)\,y'^2(x,\lambda)\,dx}-a_0c_0\right)-a_0c_1,$$

that by virtue of (34) is contradiction. Lemma 6 is proved.

**Remark 3.** *It is easy to note, that if* $\lambda<0$ *or* $\lambda=0$ *and* $\beta\in[0,\frac{\pi}{2})$, *then* $\dfrac{Ty(l,\lambda)}{y(l,\lambda)}<0$; *besides, if* $\lambda=0$ *and* $\beta=\dfrac{\pi}{2}$, *then* $Ty(l,\lambda)=0$.

**Lemma 7.** *Let* $y(x,\lambda)$ *be a nontrivial solution of problem (1), (2.a), (2.b), (2.c). If* $\lambda\le 0$, *then* $y(x,\lambda)\ne 0$ *at* $0<x<l$; *if* $\lambda<0$ *or* $\lambda=0$, $\beta\in[0,\frac{\pi}{2})$, *then* $Ty(x,\lambda)\ne 0$ *at* $0<x<l$.

**Proof.** Let $\theta(x,\lambda)$ be corresponding function from (4), where either $\lambda<0$, or $\lambda=0$ and $\beta\in[0,\frac{\pi}{2})$. From (24) it follows, that $\theta(0,\lambda)=\beta-\dfrac{\pi}{2}\in\left[-\dfrac{\pi}{2},0\right]$.

Let $\lambda = 0$ and $\beta \in [0, \frac{\pi}{2})$. By virtue of (1) we have $Ty(x,0) \equiv const$ $(0 \leq x \leq l)$. As on the base of remark 3 it is true the $y(l,0)\,Ty(l,0) < 0$, then it is obvious, that $Ty(x,0) \equiv c_0 \neq 0$ $(0 \leq x \leq l)$. So, $\theta(x,0) \neq k\pi$ $(k \in Z)$ at $0 \leq x \leq l$.

Let's note, that by virtue of equality (5.a) and (5.d) the following equality is true:

$$sgn\,(y(l,0)\,Ty(l,0)) = sgn\,(\sin\theta(l,0)\cos\theta(l,0)).$$

Hence

$$\theta(l,0) \in \left(-\frac{\pi}{2}, 0\right). \tag{35}$$

Let $\lambda < 0$. Let's prove, that $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. First of all suppose, that $\beta \in [0, \frac{\pi}{2})$. From (7.c) it follows, that the function $\theta(x,\lambda)$ takes the value of the form $k\pi$ $(k \in \mathbf{Z})$ strictly decreasing and therefore

$$\theta(x,\lambda) < 0 \quad (0 < x < l).$$

Let $\theta(l,\lambda) \in (-(m_0+1)\pi, -m_0\pi)$, where $m_0$ is some fixed nonnegative integer. As $y(l,\lambda)\,Ty(l,\lambda) < 0$, then it is obvious, that it holds

$$\theta(l,\lambda) \in \left(-m_0\pi - \frac{\pi}{2}, -m_0\pi\right). \tag{36}$$

If $m_0 = 0$, then $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. Suppose, that $m_0 \geq 1$. As $\theta(l,\lambda)$ is a continuous function of $\lambda \in (-\infty, +\infty)$, then by virtue of (35) and (36) we can state the existence of the point $\lambda_0 \in (\lambda, 0)$ such that $\theta(l,\lambda_0) \in \left(-\pi, -\frac{\pi}{2}\right)$. Hence and from (5.a), (5.d) we have $y(l,\lambda_0)\,Ty(l,\lambda_0) > 0$, that contradicts to remark 3. Consequently, in the considered case

$$\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right). \tag{37}$$

It is obvious, that $\theta(l,\lambda)$ is a continuous function on $\beta \in \left[0, \frac{\pi}{2}\right]$. Since $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $\lambda < 0$ and $\beta \in [0, \frac{\pi}{2})$, then $\theta(l,\lambda)\,|_{\beta=\pi/2} = \lim\limits_{\beta \to \frac{\pi}{2}-0} \theta(l,\lambda) \in \left[-\frac{\pi}{2}, 0\right]$. Then on the base of inequality $y(l,\lambda)\,Ty(l,\lambda) < 0$ we'll obtain, that $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $\beta = \frac{\pi}{2}$.

Suppose, that the statement of lemma, relating to the function $y(x,\lambda)$ is not true and let $x_1 \in (0,l)$ be nearest point to zero, at which $y(x_1,\lambda) = 0$.

Let's consider 5 cases.

**Case 1.** Let $\lambda < 0$ and $\beta \in \left(0, \frac{\pi}{2}\right)$. On the base of Lemma 4 from (5.a) it follows, that $\theta(x_1,\lambda) = -\frac{\pi}{2}$. Under the condition $y'(x_1,\lambda) = 0$ the function $y(x,\lambda)$ is a solution of boundary-value problem (1)-(2), where $l = x_1$ and $\gamma = \delta = 0$, that contradicts to the condition $\lambda < 0$. Hence, $y'(x_1,\lambda) \neq 0$. From here and from (5.b) we'll obtain, that $\varphi(x_1,\lambda) \neq 0$. On the base of (7.c), lemma 4 and definition of the function $w(x,\lambda)$ it holds the relation $\theta'(x_1,\lambda) = -w(x_1,\lambda)\sin\varphi(x_1,\lambda) \neq 0$. Hence, $\theta'(x_1,\lambda) < 0$. As $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$, then there exists the point $x_2 \in (x_1, 1)$ such that $\theta(x_2,\lambda) = -\frac{\pi}{2}$ (it is considered, that $x_2$ is a point, having this property and closest to $x_1$). So, $y(x_1,\lambda) = y(x_2,\lambda) = 0$. Then at the some point $\xi \in (x_1,x_2)$ we have $y'(\xi,\lambda) = 0$. Let's note, that at $x \in (x_1,x_2)$ it is true the $\theta(x,\lambda) \in \left(-\pi, -\frac{\pi}{2}\right)$. From here and from relations (5.a), (5.d) we'll obtain

$$y(x,\lambda)\,Ty(x,\lambda) = r^2(x,\lambda)\sin^2\psi(x,\lambda)\cos\theta(x,\lambda)\sin\theta(x,\lambda) > 0, \qquad (38)$$

where $0 < x_1 < x < x_2 < l$.

Let's define the angle $\delta_1$ by the following way: $\delta_1 = arctg\frac{Ty(\xi,\lambda)}{y(\xi,\lambda)}$. By virtue of (38) it holds $\delta_1 \in \left(0, \frac{\pi}{2}\right)$.

It is easy to note, that the function $y(x,\lambda)$ is nontrivial solution of boundary-value problem (1)-(2), where $l = \xi$ and $\gamma = 0$, $\delta = \delta_1$. The last contradictions to the condition $\lambda < 0$.

**Case 2.** Let $\lambda = 0$ and $\beta \in \left(0, \frac{\pi}{2}\right)$. Then $Ty(x,\lambda) \equiv c_0 \neq 0$ $(0 \le x \le l)$, $\theta(0,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$, $\theta(l,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$. Hence, $\theta(x,\lambda) \in (-\pi, 0)$. Then the proof is made similarly to the proof of case 1.

**Case 3.** Let $\lambda < 0$ and $\beta = 0$. Then $\theta(0,\lambda) = -\frac{\pi}{2}$. By virtue of (37) and by virtue of the fact that $\theta(x,\lambda)$ takes the value of the form $k\pi\,(k \in \mathbf{Z})$ strictly decreasing, then it holds either

$$-\frac{\pi}{2} < \theta(x,\lambda) < 0 \qquad (0 < x < x_1), \qquad (39)$$

or inequality

$$-\pi < \theta(x,\lambda) < -\frac{\pi}{2} \qquad (0 < x < x_1). \qquad (40)$$

At fulfillment of inequality (39) the proof of the statement $y(x,\lambda) \neq 0$ $(0 < x < l)$ is made similarly to the proof of case 1.

Let (40) hold. As $y(0,\lambda) = y(x_1,\lambda) = 0$, then at the some point $\xi \in (0,x_1)$ it holds $y'(\xi,\lambda) = 0$. Besides, relation (38) will be satisfied at $x \in (0,x_1)$. Then the proof of the statement $y(x,\lambda) \neq 0$ $(0 < x < l)$ is made similarly to the proof of case 1.

**Case 4.** Let $\lambda = 0$, $\beta = 0$. Then relations $\theta(0,0) = -\frac{\pi}{2}$, $\theta(l,0) \in -\left(\frac{\pi}{2}, 0\right)$, $Ty(x,0) \equiv c_0 \neq 0$ $(0 \le x \le l)$, $\theta(x,0) \in (-\pi, 0)$ $(0 < x < l)$ are true. Then again the proof is made similarly to the proof of case 1.

**Case 5.** And now let $\lambda = 0$ and $\beta = \frac{\pi}{2}$. From (2.b) it follows, that $Ty(0,0) = 0$. By virtue of (1) we have $Ty(x,0) \equiv 0$ $(0 \le x \le l)$ We have met the similar situation by proving lemma 4 (see (20) and (21))and there it was established, that $y(x,0) \equiv$ *const* $(0 \le x \le l)$. As $y(x_1,0) = 0$, then we have $y(x,0) \equiv 0$ $(0 \le x \le l)$. We obtain the contradiction.

In cases 1-4 practically it is proved, that if $\lambda < 0$ or $\lambda = 0$, $\beta \in [0, \frac{\pi}{2})$, then $\theta(x,\lambda) \in \left(-\frac{\pi}{2}, 0\right)$ at $0 < x < l$. Hence, by virtue of (5.d) we have $Ty(x,\lambda) \neq 0$ at $x \in (0,l)$. The proof of lemma 7 completed.

Now let's prove the basic result of the present paper.

**Theorem 4.** *The eigenvalues of boundary-value problem (1)-(2) at* $\delta \in \left(\frac{\pi}{2}, \pi\right)$ *form the infinitely increasing sequence* $\{\lambda_n(\delta)\}_{n=1}^{\infty}$ *such that*

$$\lambda_1(\delta) < \lambda_2(\delta) < ... < \lambda_n(\delta) < ...,$$

*at that* $\lambda_n(\delta) > 0$ *at* $n \geq 2$. *Besides*

*a) the eigenfunction* $y_n^{\delta}(x)$, *corresponding to the eigenvalue* $\lambda_n(\delta)$ *has exactly* $(n-1)$ *simple zeros in the interval* $(0, l)$;

*b) if* $\beta \in [0, \frac{\pi}{2})$, *then the function* $Ty_n^{\delta}(x)$ *has exactly* $(n-1)$ *simple zeros in the interval* $(0, l)$;

*c) if* $\beta = \frac{\pi}{2}$, *then the function* $Ty_1^{\delta}(x)$ *has no zeros in the interval* $(0, l)$, *and the function* $Ty_n^{\delta}(x)$ $(n \geq 2)$ *has exactly* $(n-2)$ *simple zeros in the interval* $(0, l)$;

*d) if* $\beta \in [0, \frac{\pi}{2})$, *then there exists* $\delta_0 \in (\pi/2, \pi)$ *such that* $\lambda_1(\delta) > 0$ *at* $\delta \in \left(\frac{\pi}{2}, \delta_0\right)$, $\lambda_1(\delta) = 0$ *at* $\delta = \delta_0$ *and* $\lambda_1(\delta) < 0$ *at* $\delta \in (\delta_0, \pi)$;

*e) if* $\beta = \frac{\pi}{2}$,*then* $\lambda_1(\delta) < 0$.

**Proof.** Let $y(x, \lambda)$ be a nontrivial solution of problem (1), (2.a), (2.c). The function $F(\lambda) = \frac{Ty(l, \lambda)}{y(l, \lambda)}$ by virtue of lemma 5 is a strictly increasing continuous function in the interval $(-\infty, \mu_1)$. From lemma 6 and from the equality $y(1, \mu_1) = 0$ it follows, that $\lim\limits_{\lambda \to -\infty} F(\lambda) = -\infty$, $\lim\limits_{\lambda \to -\mu_1 - 0} F(\lambda) = +\infty$ and besides, this function takes each value from $(-\infty, +\infty)$ only at unique point of the interval $(-\infty, \mu_1)$. Hence, there will be found a unique value $\lambda_1(\delta) \in (-\infty, \mu_1)$, for which $\frac{Ty(l, \lambda_1(\delta))}{y(l, \lambda_1(\delta))} = ctg\ \delta$, i.e. condition (2.d) is fulfilled. It is obvious, that $\lambda_1(\delta)$ is the first eigenvalue of problem (1)-(2). At $\beta \in [0, \frac{\pi}{2})$ it is easy to remark (see remark 3), that if $ctg\delta > \frac{Ty(l,0)}{y(l,0)}$, then $\lambda_1(\delta) > 0$; if $ctg\delta = \frac{Ty(l,0)}{y(l,0)}$, then $\lambda_1(\delta) = 0$; if $ctg\delta < \frac{Ty(l,0)}{y(l,0)}$ then $\lambda_1(\delta) < 0$. Let's note that the number $\delta_0$ appearing in the formulation of theorem 4, is defined by equality $\delta_0 = arcctg \frac{Ty(l,0)}{y(l,0)}$.

Statement e) follows from the fact, that if $\beta = \frac{\pi}{2}$ and $\lambda = 0$, then $Ty(l, \lambda) = 0$ (see again remark 3).

Let $\beta \in [0, \frac{\pi}{2})$. The function $F(\lambda)$ at $\lambda \in [0, \mu_1)$ continuously increase from the negative value $\frac{Ty(l,0)}{y(l,0)}$ to $(+\infty)$. Then the equation $F(\lambda) = 0$ has unique solution $\nu_1 \in (0, \mu_1)$, which is the eigenvalue of problem (1)-(2) at $\delta = \frac{\pi}{2}$.

Let $\frac{Ty(l,0)}{y(l,0)} < ctg\delta$. Then it is true the inequality

$$0 < \lambda_1(\delta) < \nu_1 < \mu_1. \tag{41}$$

On the base of theorem 2 from (41) it follows, that $\theta(l, \lambda_1(\delta)) < \theta(l, \nu_1)$. Besides, by virtue of (26) we have $\theta(l, \nu_1) = 0$. Consecuently, $\theta(l, \lambda_1(\delta)) < 0$. It is obvious, that $\theta(l, \lambda_1(\delta)) > -\frac{\pi}{2}$. Really, otherwise for some $\lambda^* \in [\lambda_1(\delta), \mu_1)$ the equality $\theta(l, \lambda^*) = -\frac{\pi}{2}$ would be true and $\lambda^*$ would be an eigenvalue of boundary-value problem (1)-(2) at $\delta = 0$, that is contradiction. So,

$$-\frac{\pi}{2} < \theta(l, \lambda_1(\delta)) < 0. \tag{42}$$

It is known (see theorem 5.1 and 5.2 from [1]), that if $\lambda > 0$, that the function $\theta(x, \lambda)$ takes value of the form $\frac{k\pi}{2}$ $(k \in Z)$ only strictly increasing. Hence, from (42) it follows, that $-\frac{\pi}{2} < \theta(x, \lambda_1(\delta)) < 0$ at $0 < x < l$. The last is equivalent to that the functions $y_1^\delta(x) = y(x, \lambda_1(\delta))$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0, l)$.

As was proved above, if $ctg\delta = \frac{Ty(l,0)}{u(l,0)}$, then $\lambda_1(\delta) = 0$; if $ctg\delta < \frac{Ty(l,0)}{u(l,0)}$, then $\lambda_1(\delta) < 0$.Then on the bases of lemma 7 the functions $y_1^\delta(x)$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0, l)$.

In case $\beta = \frac{\pi}{2}$ we have $\lambda_1(\delta) < 0$. Consequently again by lemma 7 the functions $y_1^\delta(x)$ and $Ty_1^\delta(x)$ have no zeros in the interval $(0, l)$.

The function $F(\lambda)$ is strictly increasing continuous function in the interval $(\mu_k, \mu_{k+1})$, where $k$ is a fixed natural number. As above, it is easy to be convinced, that there exists the unique value $\lambda_{k+1}(\delta) \in (\mu_k, \mu_{k+1})$, for which $0 > \frac{Ty(l, \lambda_{k+1}(\delta))}{y(l, \lambda_{k+1}(\delta))} = ctg\delta$. It is obvious, that $\lambda_{k+1}(\delta)$ is the $(k+1)$ the eigenvalue of problem (1)-(2).

In the interval $(\mu_k, \mu_{k+1})$ the equation $F(\lambda) = 0$ has a unique solution $\nu_{k+1} = \mu_{k+1}\left(\frac{\pi}{2}\right)$, where

$$\mu_k < \lambda_{k+1}(\delta) < \nu_{k+1} < \mu_{k+1}. \tag{43}$$

On the base of theorem 2 from (43) it follows the inequality

$$\theta(l, \mu_k) < \theta(l, \lambda_{k+1}(\delta)) < \theta(l, \nu_{k+1}). \tag{44}$$

Hence, by virtue of (26) from (44) we'll obtain

$$(2k-1)\frac{\pi}{2} < \theta(l, \lambda_{k+1}(\delta)) < 2k\frac{\pi}{2}. \tag{45}$$

As above, using theorems 5.1., 5.2 from [1] and equalities (24), (25), it is easy conclude, that at $x \in (0, l)$ it holds

$$-\frac{\pi}{2} < \theta(x, \lambda_{k+1}(\delta)) < 2k\frac{\pi}{2}$$

and the function $\theta(x, \lambda_{k+1})$ in turn takes the values of the form $\frac{m\pi}{2}$ $(m = 1, 2, ..., 2k)$ at increasing of the argument $x \in (0, l)$. It is obvious, that the eigenfunction $y_{k+1}^\delta(x)$ corresponding to the eigenvalue $\lambda_{k+1}(\delta)$, in the interval $(0, l)$ has $k$ simple zeros; at the $\beta \in [0, \frac{\pi}{2})$ function $Ty_{k+1}^\delta(x)$ has $k$ simple zeros in the interval $(0, l)$; at $\beta = \frac{\pi}{2}$ the function $Ty_{k+1}^\delta(x)$ has $(k-1)$ simple zeros in the interval $(0, l)$. Theorem 4 is proved.

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