Asymptotics of eigenvalues and eigenfunctions of a discontinuous boundary value problem

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Proceedings of IMM of NAS of Azerbaijan, 2011, vol. XXXV (XLIII), pp. 21-32. _ 21

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ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF A DISCONTINUOUS BOUNDARY VALUE PROBLEM

Abstract

The boundary problem is considered which occurs in the theory of small transversal vibrations of a smooth inhomogeneous string. The ends of the string assumed to be fixed and the midpoint of the string is damped by a pointwise force. The asymptotic behavior of eigenvalues and eigenfunctions of the considered boundary value problem is studied.

Consider the following boundary value problem

$$-y'' + q(x)y = \lambda y, \quad x \in (-1,0) \cup (0,1), \tag{1}$$

$$\left.\begin{array}{l} y(-1) = y(1) = 0 \\ y(-0) = y(+0) \\ y'(-0) - y'(+0) = \lambda m y(0). \end{array}\right\} \tag{2}$$

For the case $q(x) \equiv 0$ the asyptotics of eigenvalues and eigenfunctions, also the basis properties of eigenfunctions were investigated completely in [1].

We denote $\lambda = \rho^2$, $\operatorname{Im}\rho = \tau$. Suppose that $q(x)$ is a complex valued summable function on $(-1,1)$. Denote by $y_1(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_1(-1) = 0 \\ y_1'(-1) = \rho \end{array}\right\} \tag{3}$$

and by $y_2(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_2(1) = 0 \\ y_2'(1) = \rho \end{array}\right\} \tag{4}$$

Lemma 1. *The following integral representations hold:*

$$y_1(x,\lambda) = \sin\rho(1+x) + \frac{1}{\rho}\int_{-1}^{x} \sin\rho(x-t)q(t)y_1(t,\lambda)dt, \quad -1 < x < 0 \tag{5}$$

$$y_1'(x,\lambda) = \rho\cos\rho(1+x) + \int_{-1}^{x} \cos\rho(x-t)q(t)y_1(t,\lambda)dt, \quad -1 < x < 0 \tag{6}$$

$$y_2(x,\lambda) = \sin\rho(1-x) + \frac{1}{\rho}\int_{x}^{1} \sin\rho(t-x)q(t)y_2(t,\lambda)dt, \quad 0 < x < 1 \tag{7}$$

$$y_2'(x,\lambda) = \rho\cos\rho(1-x) - \int_{x}^{1} \cos\rho(t-x)q(t)y_2(t,\lambda)dt, \quad 0 < x < 1. \tag{8}$$

Proof. Since $y_1(x,\lambda)$ satisfies (1) on $(-1,0)$, then

$$\int_{-1}^{x} \sin\rho(x-t)q(t)y_1(t)dt = \int_{-1}^{x} \sin\rho(x-t)y_1''(t,\lambda)dt + \rho^2 \int_{-1}^{x} \sin\rho(x-t)y_1(t,\lambda)dt.$$

Integrating by part the first integral in the right-hand side of the last equation twice and taking into account (3), we find

$$\int_{-1}^{x} \sin\rho(x-t)q(t)y_1(t)dt = -\rho\sin\rho(x+1) + \rho y_1(x,\lambda),$$

i.e. the equality (5).

The equality (6) is obtained by differentiating the equality (5).

The equalities (7) and (8) are obtained similarly.

Lemma 2. *The following asymptotic formulas hold when* $|\rho| \to \infty$

$$y_1(x,\lambda) = O(e^{|\tau|(1+x)}), \tag{9}$$

$$y_2(x,\lambda) = O(e^{|\tau|(1-x)}), \tag{10}$$

more precisely

$$y_1(x,\lambda) = \sin\rho(1+x) + O\left(\frac{e^{|\tau|(1+x)}}{|\rho|}\right), \tag{11}$$

$$y_2(x,\lambda) = \sin\rho(1-x) + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \tag{12}$$

All estimates are satisfied uniformly on x for $y_1(x,\lambda)$ when $-1 \le x \le 0$ and for $y_2(x,\lambda)$ when $0 \le x \le 1$.

The proof repeats that lemma in [2] word for word.

Theorem 1. *Let $q(x)$ is a complex valued function summable on $[-1,1]$ and let*

$$d = 4 + (mq_2(0))^2 + (mq_1(0))^2 + 8mq_2(0) - 2m^2q_2(0)q_1(0) \neq 0,$$

here

$$q_1(0) = \frac{1}{2}\int_{-1}^{0} q(t)dt$$

and

$$q_2(0) = \frac{1}{2}\int_{0}^{1} q(t)dt.$$

Then the spectrum of problem (1)-(2) consists of two sequences $\lambda_{1,n} = \rho_{1,n}^2$, $n = 1,2,...$ and $\lambda_{2,n} = \rho_{2,n}^2$, $n = 1,2,...$ of asymptotically simple eigenvalues, where $\rho_{1,n}$ and $\rho_{2,n}$ holds the following asymptotic equalities

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)$$

and

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right),$$

here α_1 and α_2 are different numbers and are defined by the following way:

$$\alpha_1 = \frac{-(2mq_2(0) + mq_1(0)) + \sqrt{d}}{-2m\pi},$$

$$\alpha_2 = \frac{-(2mq_2(0) + mq_1(0)) - \sqrt{d}}{-2m\pi},$$

$$0 \le \arg\sqrt{d} < \pi.$$

Proof. Substitute asymptotics for $y_1(x)$ from (11) in the right-hand side of (5):

$$y_1(x) = \sin\rho(1+x) + \frac{1}{\rho}\int_{-1}^{x} \sin\rho(x-t)q(t)\left[\sin\rho(1+t) + O\left(\frac{e^{|\tau|(1+t)}}{|\rho|}\right)\right]dt =$$

$$= \sin\rho(1+x) + \frac{1}{\rho}\int_{-1}^{x} \sin\rho(x-t)\sin\rho(1+t)q(t)dt +$$

$$+\frac{1}{\rho^2}\int_{-1}^{x} \sin\rho(x-t)q(t)O\left(e^{|\tau|(1+t)}\right) = \sin\rho(1+x) +$$

$$+\frac{1}{2\rho}\int_{-1}^{x}\left[\cos\rho(x-2t-1) - \cos\rho(1+x)\right]q(t)dt +$$

$$+\frac{1}{\rho^2}\int_{-1}^{x} \sin\rho(x-t)q(t)O\left(e^{|\tau|(1+t)}\right)dt = \sin\rho(1+x) -$$

$$-\frac{1}{2\rho}\cos\rho(1+x)\int_{-1}^{x} q(t)dt + \frac{1}{2\rho}\int_{-1}^{x}\cos\rho(x-2t-1)q(t)\,dt +$$

$$+\frac{1}{\rho^2}\int_{-1}^{x} \sin\rho(x-t)q(t)O\left(e^{|\tau|(1+t)}\right)dt = \sin\rho(1+x) -$$

$$-\frac{1}{\rho}\cos\rho(1+x)\left(\frac{1}{2}\int_{-1}^{x} q(t)dt\right) + \frac{1}{2\rho}\int_{-1}^{x}\cos\rho(x-2t-1)q(t)dt +$$

$$+\frac{e^{|\tau|(1+x)}}{\rho^2}\int_{-1}^{x}\frac{\sin\rho(x-t)}{e^{|\tau|(x-t)}}O(1)q(t)dt.$$

Hence,

$$y_1(x) = \sin\rho(1+x) - \frac{1}{\rho}q_1(x)\cos\rho(1+x)q_1(x) +$$

$$+\frac{1}{2\rho}\int_{-1}^{x}\cos\rho(x-2t-1)q(t)dt + O\left(\frac{e^{|\tau|(1+x)}}{|\rho|^2}\right), \tag{13}$$

here for $-1 \le x \le 0$

$$q_1(x) = \frac{1}{2}\int_{-1}^{x} q(t)dt.$$

Substitute asymptotics for $y_1(x)$ from (11) in the right-hand side of (6):

$$y_1'(x) = \rho\cos\rho(1+x) + \int_{-1}^{x}\cos\rho(x-t)q(t)\left[\sin\rho(1+t) + O\left(\frac{e^{|\tau|(1+t)}}{|\rho|}\right)\right]dt =$$

$$= \rho\cos\rho(1+x) + \int_{-1}^{x}\cos\rho(x-t)\sin\rho(1+t)q(t)dt+$$

$$+\frac{1}{\rho}\int_{-1}^{x}\cos\rho(x-t)q(t)O\left(e^{|\tau|(1+t)}\right)dt = \rho\cos\rho(1+x)+$$

$$+\frac{1}{2}\int_{-1}^{x}\left[\sin\rho(1+x)+\sin\rho(1-x+2t)\right]q(t)dt+$$

$$+\frac{1}{\rho}\int_{-1}^{x}\frac{\cos\rho(x-t)}{e^{|\tau|(x-t)}}O(1)q(t)dt\cdot e^{|\tau|(1+x)}.$$

Hence

$$y_1'(x) = \rho\cos\rho(1+x) + \frac{1}{2}\sin\rho(1+x)\int_{-1}^{x}q(t)dt+$$

$$+\frac{1}{2}\int_{-1}^{x}q(t)\sin\rho(1+2t-x)dt + O\left(\frac{e^{|\tau|(1+x)}}{|\rho|}\right). \tag{14}$$

The following asymptotic equalities are obtained analogously:

$$y_2(x) = \sin\rho(1-x) - \frac{1}{\rho}q_2(x)\cos\rho(1-x)q_2(x)+$$

$$+\frac{1}{2\rho}\int_{x}^{1}\cos\rho(2t-x-1)q(t)dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|^2}\right) \tag{15}$$

and

$$y_2'(x) = -\rho\cos\rho(1-x) - q_2(x)\sin\rho(1-x)-$$

$$-\frac{1}{2}\int_{x}^{1}q(t)\sin\rho(1+x-2t)dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right), \tag{16}$$

here for $0 \le x \le 1$

$$q_2(x) = \frac{1}{2}\int_{x}^{1}q(t)dt.$$

Obviously, for any $\lambda \neq 0$ the solution $y(x,\lambda)$ of the problem (1)-(2) have to be in the form

$$y(x) = \begin{cases} C_1 y_1(x), & \text{for } \ -1 < x < 0, \\ C_2 y_2(x), & \text{for } \quad 0 < x < 1, \end{cases}$$

here C_1 and C_2 are complex numbers. $\lambda \neq 0$ is an eigenvalue of the problem (1)-(2) if and only if C_1 and C_2 are nontrivial solutions of following homogeneous system of linear equations:

$$\begin{cases} C_1\left(\sin\rho-\dfrac{q_1(0)}{\rho}\cos\rho+\dfrac{1}{2\rho}\int\limits_{-1}^{0}\cos\rho(1+2t)q(t)dt+O\left(\dfrac{e^{|\tau|}}{|\rho|^2}\right)\right)- \\ -C_2\left(\sin\rho-\dfrac{q_2(0)}{\rho}\cos\rho+\dfrac{1}{2\rho}\int\limits_{0}^{1}\cos\rho(2t-1)q(t)dt+O\left(\dfrac{e^{|\tau|}}{|\rho|^2}\right)\right)=0 \\ C_1\left(\rho\cos\rho+q_1(0)\sin\rho+\dfrac{1}{2}\int\limits_{-1}^{0}q(t)\sin\rho(1+2t)dt+O\left(\dfrac{e^{|\tau|}}{|\rho|}\right)\right)- \\ -C_2\left(-\rho\cos\rho-q_2(0)\sin\rho-\dfrac{1}{2}\int\limits_{0}^{1}q(t)\sin\rho(1-2t)dt+O\left(\dfrac{e^{|\tau|}}{|\rho|}\right)\right)= \\ =C_1\rho^2 m\left(\sin\rho-\dfrac{q_1(0)}{\rho}\cos\rho+\dfrac{1}{2\rho}\int\limits_{-1}^{0}\cos\rho(1+2t)q(t)dt+O\left(\dfrac{e^{|\tau|}}{|\rho|^2}\right)\right) \end{cases}$$

To define eigenvalues we obtain following equation

$$\triangle(\lambda)=\begin{vmatrix} A_{11}(\rho) & A_{12}(\rho) \\ A_{21}(\rho) & A_{22}(\rho) \end{vmatrix}=0,$$

here

$$A_{11}(\rho)=\sin\rho-\frac{q_1(0)}{\rho}\cos\rho+\frac{1}{2\rho}\int\limits_{-1}^{0}\cos\rho(1+2t)q(t)dt+O\left(\frac{e^{|\tau|}}{|\rho|^2}\right),$$

$$A_{12}(\rho)=-\sin\rho+\frac{q_2(0)}{\rho}\cos\rho-\frac{1}{2\rho}\int\limits_{0}^{1}\cos\rho(2t-1)q(t)dt+O\left(\frac{e^{|\tau|}}{|\rho|^2}\right),$$

$$A_{21}(\rho)=(\rho\cos\rho-\rho^2 m\sin\rho)+(q_1(0)\sin\rho+\rho m q_1(0)\cos\rho)+$$
$$+\frac{1}{2}\int\limits_{-1}^{0}q(t)\sin\rho(1+2t)dt-\frac{1}{2}m\rho\int\limits_{-1}^{0}\cos\rho(1+2t)q(t)dt+O(e^{|\tau|}),$$

$$A_{22}(\rho)=\rho\cos\rho+q_2(0)\sin\rho+\frac{1}{2}\int\limits_{0}^{1}q(t)\sin\rho(1-2t)dt+O\left(\frac{e^{|\tau|}}{|\rho|}\right).$$

Using that, for any complex number z

$$|\sin z|\le e^{|\operatorname{Im} z|}$$

and

$$|\cos z|\le e^{|\operatorname{Im} z|},$$

we can write

$$|\cos\rho(2t-1)|\le e^{|\tau|},\quad \text{for}\quad 0\le t\le 1,$$
$$|\sin\rho(1+2t)|\le e^{|\tau|},\quad \text{for}\quad -1\le t\le 0,$$
$$|\cos\rho(1+2t)|\le e^{|\tau|},\quad \text{for}\quad -1\le t\le 0,$$
$$|\sin\rho(1-2t)|\le e^{|\tau|},\quad \text{for}\quad 0\le t\le 1.$$

Taking into account the last inequalities, for $|\rho| \to \infty$ we obtain:

$$\int_{-1}^{0} q(t)\cos\rho(1+2t)dt = O(e^{|\tau|}),$$

$$\int_{0}^{1} q(t)\cos\rho(2t-1)dt = O(e^{|\tau|}),$$

$$\int_{-1}^{0} q(t)\sin\rho(1+2t) = O(e^{|\tau|}),$$

$$\int_{0}^{1} q(t)\sin\rho(1-2t)dt = O(e^{|\tau|}).$$

From the last asymptotic formulas we obtain that $\triangle(\lambda)$ can be written as the form:

$$\triangle(\lambda) = \begin{vmatrix} \sin\rho & -\sin\rho \\ \rho\cos\rho - \rho^2 m\sin\rho & \rho\cos\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\rho & \dfrac{q_2(0)}{\rho}\cos\rho \\ \rho\cos\rho - \rho^2 m\sin\rho & q_2(0)\sin\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\rho & \dfrac{1}{2\rho}\int_0^1 \cos\rho(2t-1)q(t)dt \\ \rho\cos\rho - \rho^2 m\sin\rho & \dfrac{1}{2}\int_0^1 q(t)\sin\rho(1-2t)dt \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\rho & O\left(\dfrac{e^{|\tau|}}{|\rho|^2}\right) \\ \rho\cos\rho - \rho^2 m\sin\rho & O\left(\dfrac{e^{|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+ \begin{vmatrix} -\dfrac{q_1(0)}{\rho}\cos\rho & -\sin\rho \\ q_1(0)\sin\rho + m\rho q_1(0)\cos\rho & \rho\cos\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} -\dfrac{q_1(0)}{\rho}\cos\rho & \dfrac{q_2(0)}{\rho}\cos\rho \\ q_1(0)\sin\rho + m\rho q_1(0)\cos\rho & q_2(0)\sin\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} -\dfrac{q_1(0)}{\rho}\cos\rho & -\dfrac{1}{2\rho}\int_0^1 \cos\rho(2t-1)q(t)dt \\ q_1(0)\sin\rho + m\rho q_1(0)\cos\rho & \dfrac{1}{2}\int_0^1 q(t)\sin\rho(1-2t)dt \end{vmatrix} +$$

$$+ \begin{vmatrix} -\dfrac{q_1(0)}{\rho}\cos\rho & O\left(\dfrac{e^{|\tau|}}{|\rho|^2}\right) \\ q_1(0)\sin\rho + m\rho q_1(0)\cos\rho & O\left(\dfrac{e^{|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt & -\sin\rho \\ \frac{1}{2}\int\limits_{-1}^{0} q(t)\sin\rho(1+2t)dt-\frac{1}{2}m\rho\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt & \rho\cos\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt & \frac{q_2(0)}{\rho}\cos\rho \\ \frac{1}{2}\int\limits_{-1}^{0} q(t)\sin\rho(1+2t)dt-\frac{1}{2}m\rho\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt & q_2(0)\sin\rho \end{vmatrix}+$$

$$+\begin{vmatrix} O\left(\frac{e^{|\tau|}}{|\rho|^2}\right) & -\sin\rho \\ O(e^{|\tau|}) & \rho\cos\rho \end{vmatrix}+O\left(\frac{e^{2|\tau|}}{|\rho|}\right)$$

Opening all determinants in the last equality, we obtain the following for the function $\triangle(\lambda)$:

$$\triangle(\lambda)=\left[-m\rho^2+q_2(0)+q_1(0)\right]\sin^2\rho+$$
$$+\left(2\rho+m\rho q_2(0)+m\rho q_1(0)-2\frac{q_1(0)q_2(0)}{\rho}\right)\sin\rho\cos\rho+$$
$$+\left[\frac{1}{2}\int\limits_0^1 q(t)\sin\rho(1-2t)dt+\frac{1}{2}m\rho\int\limits_0^1\cos\rho(2t-1)q(t)dt+\ O\left(\frac{e^{|\tau|}}{|\rho|}\right)+\right.$$
$$+O\left(e^{|\tau|}\right)-q_1(0)\frac{1}{2\rho}\int\limits_0^1\cos\rho(2t-1)q(t)dt+\ O\left(\frac{e^{|\tau|}}{|\rho|^2}\right)+\frac{1}{2}\int\limits_{-1}^{0} q(t)\sin\rho(1+2t)dt-$$
$$\left.-\frac{1}{2}m\rho\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt+\frac{q_2(0)}{2}\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt\right]\sin\rho+$$
$$+\left[-\frac{1}{2}\int\limits_0^1\cos\rho(2t-1)q(t)dt+O\left(\frac{e^{|\tau|}}{|\rho|}\right)-\frac{q_1(0)}{2\rho}\int\limits_0^1 q(t)\sin\rho(1-2t)dt-\right.$$
$$-\frac{mq_1(0)}{2}\int\limits_0^1\cos\rho(2t-1)q(t)dt+O\left(\frac{e^{|\tau|}}{\rho}\right)+O\left(\frac{e^{|\tau|}}{\rho}\right)+$$
$$+\frac{1}{2}\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt-\frac{q_2(0)}{2\rho}\int\limits_{-1}^{0} q(t)\sin\rho(1+2t)dt+$$
$$\left.+\frac{q_2(0)}{2}m\int\limits_{-1}^{0} q(t)\cos\rho(1+2t)dt\right]\cos\rho+$$
$$+\left[q_2(0)-q_1(0)-mq_2(0)q_1(0)\right]\cos^2\rho+O\left(\frac{e^{2|\tau|}}{\rho}\right). \tag{17}$$

Circle the points $\widetilde{\rho}_k=\pi k,\ \ k=1,2,...$ by the circles with radius $\frac{\pi}{4}$. Out of these circles the inequality

$$|\delta(\rho)|\geq C\,|\rho|^2\,e^{2|\tau|}$$

holds for the function

$$\delta(\rho)=\left[-\rho^2+q_2(0)+q_1(0)\right]\sin^2\rho,$$

here $C>0$ is a constant. Since modules of remained summands of the right-hand side of equality (17) don't exceed $A|\rho|e^{2|\tau|}$ (here $A>0$ is a constant), then by Rouchet theorem for sufficiently large k function $\triangle(\lambda)$ posseses exactly two zeroes multiplicity taking into account in the $\frac{\pi}{4}$ neighborhood of every point πk. Hence, all zeroes of $\triangle(\lambda)$ lie in some horizontal strip $|\operatorname{Im}\rho|\le\alpha$, here α is a positive constant.

Since, all zeroes of $\triangle(\rho^2)$ belong to strip $|\operatorname{Im}\rho|\le\alpha$ in sequel assume that ρ runs only in this strip. Under this assumptation the following asymptotic equalities are true for $|\rho|\to+\infty$:

$$\left.\begin{array}{c} O\left(\dfrac{e^{|\tau|}}{\rho}\right)=O\left(\dfrac{e^{2|\tau|}}{\rho}\right)=O\left(\dfrac{1}{\rho}\right),\\ O\left(\dfrac{e^{|\tau|}}{\rho^2}\right)=O\left(\dfrac{1}{\rho^2}\right),\\ O\left(e^{|\tau|}\right)=O(1). \end{array}\right\}\qquad(18)$$

In the other hand, in the strip $|\operatorname{Im}\rho|\le\alpha$

$$\int\limits_{-1}^{0}q(t)\cos\rho(1+2t)dt=\int\limits_{0}^{1}q(t)\cos(2t-1)dt=\int\limits_{-1}^{0}q(t)\sin\rho(1+2t)dt=$$

$$=\int\limits_{0}^{1}q(t)\sin\rho(1-2t)dt=o(1)\qquad(19)$$

for $|\rho|\to+\infty$.

In the discs centrated at the points $\widetilde{\rho}_k=\pi k,\quad k=1,2,...,$ with radius $\frac{\pi}{4}$ the inequality

$$\left|\frac{1}{\cos\rho}\right|\le M,\qquad(20)$$

holds, where M is a constant not depend on ρ.

Taking into account (17)-(20) the equation $\triangle(\rho^2)=0$ can be written in the form:

$$\left[-m\rho^2+q_2(0)+q_1(0)\right]tg^2\rho+\left[2\rho+m\rho q_2(0)+m\rho q_1(0)+o(\rho)\right]tg\rho+$$

$$+\left[q_2(0)-q_1(0)-mq_2(0)q_1(0)+o(1)\right]=0.\qquad(21)$$

The prime part of the discriminant of this quadratic equation (by $tg\rho$) is

$$\left[4+(mq_2(0))^2+8mq_2(0)-2m^2q_2(0)q_1(0)\right]\rho^2=d\rho^2,$$

here

$$d=4+(mq_2(0))^2+8mq_2(0)-2m^2q_2(0)q_1(0)$$

The set of roots of (21) are the union of roots of equations

$$tg\rho=\frac{-(2+mq_2(0)+mq_1(0)+o(1))\rho+\sqrt{d+o(1)}\rho}{2\left[-m\rho^2+q_2(0)+q_1(0)\right]}$$

and

$$tg\rho = \frac{-(2 + mq_2(0) + mq_1(0) + o(1))\rho - \sqrt{d + o(1)}\rho}{-2\left[-m\rho^2 + q_2(0) + q_1(0)\right]}.$$

We write these equations in the form

$$tg\rho = \frac{a_1}{\rho} + o\left(\frac{1}{\rho}\right) \tag{22}$$

and

$$tg\rho = \frac{a_2}{\rho} + o\left(\frac{1}{\rho}\right), \tag{23}$$

correspondingly, here

$$a_1 = \frac{-(2 + mq_2(0) + mq_1(0)) + \sqrt{d}}{-2m}$$

and

$$a_2 = \frac{-(2 + mq_2(0) + mq_1(0)) - \sqrt{d}}{-2m},$$

and $\sqrt{d}$ is the number satisfying $0 \le \arg\sqrt{d} < \pi$. By the condition of the Theorem 1 $d \neq 0$, we obtain that $a_1 \neq a_2$. Consider the equation (22). From the Rouche theorem we obtain that the roots $\rho_{1,n}$ of this equation are asymptoticaly simple and are in the form

$$\rho_{1,n} = \pi n + \varepsilon_n,$$

here $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Taking this into account in (22), we obtain

$$tg(\pi n + \varepsilon_n) = \frac{a_1}{\pi n + \varepsilon_n} + o\left(\frac{1}{n}\right),$$

or

$$tg\varepsilon_n = \frac{\frac{a_1}{\pi}}{n} + o\left(\frac{1}{n}\right).$$

From here we have

$$\varepsilon_n = \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right),$$

here $\alpha_1 = \dfrac{a_1}{\pi}$.

Hence, for the sequence $\rho_{1,n}$ of roots of the equation (22) we obtained following asymptotic formula

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right).$$

By the same way we obtain the asymptoticaly simpleness and following asymptotic formula for the sequence $\rho_{2,n}$ of roots of equation (23):

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right)$$

here $\alpha_2 = \dfrac{a_2}{\pi}$. By the theorem $d \neq 0$, then $\alpha_1 \neq \alpha_2$.

Theorem is proved.

Now, let's pass to study the asymptotic behavior of eigenfunctions of the problem (1)-(2). From the asymptotic equalities obtained for $\rho_{1,n}$ and $\rho_{2,n}$ and asymptotic expression for $A_{22}(\rho)$ for the sufficiently large n we have

$$A_{22}(\rho_{1,n}) \neq 0 \quad \text{and} \quad A_{22}(\rho_{2,n}) \neq 0.$$

Hence, for the sufficiently large n the eigenfunction of the problem (1)-(2) corresponding to eigenvalue $\lambda_{1,n} = (\rho_{1,n})^2$ will be

$$y_{1,n}(x) = \begin{cases} \dfrac{1}{\rho_{1,n}} A_{22}(\rho_{1,n}) y_1(x, \lambda_{1,n}), & \text{for} \quad x \in [-1, 0] \\ -\dfrac{1}{\rho_{1,n}} A_{21}(\rho_{1,n}) y_2(x, \lambda_{1,n}), & \text{for} \quad x \in [1, 0], \end{cases}$$

and the eigenfunction corresponding to eigenvalue $\lambda_{2,n} = (\rho_{2,n})^2$ will be

$$y_{2,n}(x) = \begin{cases} \dfrac{1}{\rho_{2,n}} A_{22}(\rho_{2,n}) y_1(x, \lambda_{2,n}), & \text{for} \quad x \in [-1, 0] \\ -\dfrac{1}{\rho_{2,n}} A_{21}(\rho_{2,n}) y_2(x, \lambda_{2,n}), & \text{for} \quad x \in [-1, 0]. \end{cases}$$

Let $x \in [-1, 0]$. Since,

$$\left. \begin{array}{ll} \cos z = 1 + O(z^2), & z \to 0 \\ \sin z = z + O(z^3) = O(z), & z \to 0 \end{array} \right\},$$

then we have:

$$\frac{1}{\rho_{1,n}} A_{22}(\rho_{1,n}) = \cos\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) =$$

$$= (-1)^n \cos\left(\frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) =$$

$$= (-1)^n \left[1 + O\left(\frac{1}{n^2}\right)\right] + O\left(\frac{1}{n}\right) = (-1)^n + O\left(\frac{1}{n}\right).$$

From (11) we have

$$y_1(x, \lambda_{1,}) = \sin \rho_{1,n}(1 + x) + O\left(\frac{1}{n}\right).$$

In the other hand

$$\sin \rho_{1,n}(1 + x) = \sin\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right)(1 + x) =$$

$$= \sin\left(\pi n + \pi n x + O\left(\frac{1}{n}\right)\right) = (-1)^n \sin\left(\pi n x + O\left(\frac{1}{n}\right)\right) =$$

$$= (-1)^n \left[\sin \pi n x + O\left(\frac{1}{n}\right)\right]$$

Finally, for $x \in [-1, 0]$ we have

$$y_{1,n}(x) = \left((-1)^n + O\left(\frac{1}{n}\right)\right)\left((-1)^n \sin \pi n x + O\left(\frac{1}{n}\right)\right) = \sin \pi n x + O\left(\frac{1}{n}\right).$$

Now, let $x \in [0, 1]$. We have:

$$-\frac{1}{\rho_{1,n}} A_{21}(\rho_{1,n}) = -\cos \rho_{1,n} + m\rho_{1,n} \sin \rho_{1,n} - q_1(0) m \cos \rho_{1,n} + o(1) =$$

$$= -(1 + mq_1(0)) \cos\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + m\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) \times$$

$$\times \sin\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + o(1) = (1 + mq_1(0))(-1)^{n+1} \cos\left(\frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) -$$

$$-m(-1)^{n+1}\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) \times \sin\left(\frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + o(1) =$$

$$= (1 + mq_1(0))(-1)^{n+1}\left(1 + O\left(\frac{1}{n^2}\right)\right) - m(-1)^{n+1}\left(\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) \times$$

$$\times \left(\frac{\alpha_1}{n} + o\left(\frac{1}{n} + O\left(\frac{1}{n^3}\right)\right)\right) + o(1) = (-1)^{n+1}(1 + mq_1(0) - m\alpha_1\pi) + o(1)$$

By the same way as for $y_1(x, \lambda_{1,n})$ we can prove, that

$$y_2(x, \lambda_{1,n}) = (-1)^{n+1} \sin \pi n x + O\left(\frac{1}{n}\right).$$

Finally, for $x \in [0, 1]$ we have

$$y_{1,n}(x) = \gamma_{1,n} \sin \pi n x + O\left(\frac{1}{n}\right)$$

here

$$\gamma_{1,n} = 1 + mq_1(0) - m\alpha_1\pi + o\,(1)\,. \tag{24}$$

The following asymptotic equality for the eigenfunction $y_{2,n}(x)$ proves analogously:

$$y_{2,n}(x) = \begin{cases} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1, 0] \\ \gamma_{2,n} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1, 0] \end{cases}$$

here

$$\gamma_{2,n}(x) = 1 + mq_1(0) - m\alpha_2\pi + o\,(1) \tag{25}$$

We proved the following

Theorem 2. *Let the function $q(x)$ satisfies the conditions of the Theorem 1. Then the eigenfunctions $y_{1,n}(x)$ coressponding to eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2$ and the eigenfunctions $y_{2,n}(x)$ corresponding to eigenvalues $\lambda_{2,n} = (\rho_{2,n})^2$ satisfies the following asymptotic equalyties:*

$$y_{1,n}(x) = \begin{cases} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1.0] \\ \gamma_{1,n} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [0, 1] \end{cases} \tag{26}$$

$$y_{2,n}(x)=\begin{cases}\sin\pi nx+O\left(\frac{1}{n}\right), & x\in[-1.0]\\ \gamma_{2,n}\sin\pi nx+O\left(\frac{1}{n}\right), & x\in[0,1]\end{cases} \tag{27}$$

here $\gamma_{1,n}$ *and* $\gamma_{2,n}$ *holds (24) and (25) respectively.*

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Received July 05, 2011; Revised October 12, 2011