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Approximation of analytic functions in multiply connected domains by linear operators

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Let $D_0 = \{z \in C : \ |z - a_0| < R\}$, $a_1, ..., a_m \in D_0$ and $D = \{z \in D_0 : , |z - a_i| > r_i,$ $i = \overline{1, m}\}$, where $0 < r_i < R - |a_0 - a_i|$, $i = \overline{1, m}$, $r_i + r_j < |a_i - a_j|$, $i, j = \overline{1, m}$, $i \neq j$. We will denote by $A\,(D)$ the space of all analytic functions in $m + 1$ connected domain D with the topology of compact convergence. This means that, by a convergence in this space we will mean the uniform convergence in any compact of D. For $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r'_i < R' - |a_0 - a_i|$, $i = \overline{1, m}$, $r'_i + r'_j < |a_i - a_j|$, $i, j = \overline{1, m}$, $i \neq j$ we will consider the seminorms

$$\|f\|_{A(D), r'_1, ..., r'_m, R'} = \max \left\{ |f\,(z)| \ : \ z \in \bar{D}_{r'_1, ..., \, r'_m, \, R'} \right\},$$

that convert $A\,(D)$ into a Frechet–type space, where

$$\bar{D}_{r'_1, ..., r'_m, R'} = \left\{ z \in C : \ |z - a_0| \le R', \ |z - a_i| \ge r'_i, \, i = \overline{1, \, m} \right\}.$$

It is known [see, for example, 2] that the system of functions 1, $(z - a_0)^k$, $(z - a_1)^{-k}$,..., $(z - a_m)^{-k}$, $k \in N$, forms a basis for $A\,(D)$, i.e. every function $f \in A\,(D)$ can be uniquely represented in the form

$$f\,(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)}\,(z - a_i)^{\omega_i(k)}, \tag{1}$$

where $\omega_0\,(k) = k$, $k \in Z_+$, $\omega_i\,(k) = -k - 1$ for $i = \overline{1, \, m}$, $k \in Z_+$, and the coefficients $f_k^{(i)}$, $k \in Z_+$, $i = \overline{0, \, m}$ are defined by the formula

$$f_k^{(i)} = \frac{1}{2\pi i} \int_{\Gamma_i} f\,(z)\,(z - a_i)^{-\omega_i(k) - 1}\,dz, k \in Z_+, i = \overline{0, \, m}$$

and Γ_i, $i = \overline{0, m}$ are any circles centered at the points a_i and belonging to the domain D.

Denote $\alpha_0 = R^{-1}$ and $\alpha_i = r_i$ for $i = \overline{1, \, m}$.

First we prove the following theorem on the convergence to zero in $A\,(D)$.

Theorem 1. The sequence $f_n\,(z)$ convergence to zero in $A\,(D)$ if and only if the coefficients of expansion

$$f_n\,(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_{n,k}^{(i)}\,(z - a_i)^{\omega_i(k)}$$

satisfy the conditions

$$\left| f_{n,k}^{(i)} \right| < \varepsilon_n\,(1 + \delta_n)^k\,\alpha_i^k\,, k \in Z_+, i = \overline{0, m} \tag{2}$$

for any $n \in N$, where

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = 0. \tag{3}$$

Now we consider the linear operators in $A(D)$. It follows from (1) that for any linear operator $T : A(D) \to A(D)$ the expansion

$$(Tf)(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,\,(i,j)} f_p^{(j)} \quad (z - a_i)^{\omega_i(k)} \right.$$

is valid, where $f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)}$ and

$$T\left((z - a_j)^{\omega_j(p)} \right) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} T_{k,p,\,(i,j)} (z - a_i)^{\omega_i(k)}, p \in Z_+, j = \overline{0, m}.$$

Let the system of sequences of positive numbers $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0, m} \right\}$ satisfies the conditions:

$$\forall i = \overline{0, m}, \forall k \in Z_+ : \Delta_k^{(i)}(g) = \inf \left\{ \left| \sqrt{g_k^{(i)}} - \sqrt{g_p^{(j)}} \right| : p \in Z_+,\ j = \overline{0, m},\ (j,p) \neq (i,k) \right\} > 0, \tag{4}$$

$$\forall i = \overline{0, m} : \lim_{k\to\infty} \left(\Delta_k^{(i)}(g) \right)^{\frac{1}{k}} = 1, \lim_{k\to\infty} \left(g_k^{(i)} \right)^{\frac{1}{k}} = 1. \tag{5}$$

Definition 1. By $A_g(D)$ we denote the set of analytic functions

$$f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)} \in A(D)$$

whose coefficients satisfy the following conditions:

$$\left| f_k^{(i)} \right| \leq M_f \, g_k^{(i)} \, \alpha_i^k, k \in Z_+, i = \overline{0, m}, \tag{6}$$

where M_f is a constant independent of k.

Theorem 2. Let $T_n : A(D) \to A(D)$ be a sequence of linear operators

$$(T_n f)(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,\,(i,j)}^{(n)} f_p^{(j)} \quad (z - a_i)^{\omega_i(k)} \right., \tag{7}$$

where $f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)} \in A(D)$. If there exist sequences ε_n and δ_n satisfying (3) such that the inequalities

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,\,(i,j)}^{(n)} \alpha_j^p - \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0, m}, k \in Z_+ \tag{8}$$

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} \left| T_{k,p,\,(i,j)}^{(n)} \right| \alpha_j^p - \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0, m}, k \in Z_+, \tag{9}$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|\sqrt{g_p^{(j)}}\alpha_j^p-\sqrt{g_k^{(i)}}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+ \quad (10)$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|g_p^{(j)}\alpha_j^p-g_k^{(i)}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+ \quad (11)$$

holds, then for any function $f\in A_g(D)$ and for every $\max\limits_{i=\overline{1,m}}\left[|a_0-a_i|+r_i\right]<R'<R$, $r_i<r_i'<R'-|a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j'<|a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$ we have

$$\lim_{n\to\infty}\|T_nf-f\|_{A(D),r_1',\ldots,r_m',R'}=0.$$

Now, we will present the following general result on approximation in $A(D)$.

Theorem 3. Let the sequences of positive numbers $b=\left\{\left\{b_k^{(i)}\right\}_{k=0}^{\infty}:\ i=\overline{0,m}\right\}$ and $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty}:\ i=\overline{0,m}\right\}$ satisfy the conditions (4), (5) and $T_n:A(D)\to A(D)$ be a linear operators defined as (7). If there exist sequences ε_n and δ_n satisfying (3) such that the inequalities

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}T_{k,p,\,(i,j)}^{(n)}g_p^{(j)}\alpha_j^p-g_k^{(i)}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+, \quad (12)$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|g_p^{(j)}\alpha_j^p-g_k^{(i)}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+, \quad (13)$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|\sqrt{b_p^{(j)}}g_p^{(j)}\alpha_j^p-\sqrt{b_k^{(i)}}g_k^{(i)}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+, 18)$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|b_p^{(j)}g_p^{(j)}\alpha_j^p-b_k^{(i)}g_k^{(i)}\alpha_i^k\right|<\varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k, i=\overline{0,m}, k\in Z_+ \quad (14)$$

holds, then for any function $f\in A_g(D)$ and for every $\max\limits_{i=\overline{1,m}}\left[|a_0-a_i|+r_i\right]<R'<R$, $r_i<r_i'<R'-|a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j'<|a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$ we have

$$\lim_{n\to\infty}\|T_nf-f\|_{A(D),r_1',\ldots,r_m',R'}=0.$$

Definition 2. Linear operator $T:A(D)\to A(D)$ is called k-positive if it preserve the subclass of analytic functions with positive coefficients in the expansion of functions in series (1).

It is obvious that the k-positiveness of the operator

$$(Tf)(z)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{p=0}^{\infty}T_{k,p,\,(i,j)}f_p^{(j)}\quad(z-a_i)^{\omega_i(k)},\right.$$

where $f(z)=\ {}_{i=0}^{m}\ {}_{k=0}^{\infty}f_k^{(i)}(z-a_i)^{\omega_i(k)}\in A(D)$, is equivalent to the non-negativeness of the coefficients $T_{k,p,(i,j)}$, $i,j=\overline{0,m}$, $k,p\in Z_+$.

Let the sequences of positive numbers $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty}\ :\ i=\overline{0,m}\right\}$ satisfy the conditions (4), (5). Denote

$$h_\nu\left(z\right)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(g_k^{(i)}\right)^{\frac{\nu}{2}}\alpha_i^k\left(z-a_i\right)^{\omega_i(k)},\nu=0,\,1,\,2. \tag{15}$$

Theorem 4. Let $T_n : A\left(D\right)\to A\left(D\right)$ be a sequence of linear k-positive operators. The sequence $T_nf\left(z\right)$ tends to $f\left(z\right)$ in $A\left(D\right)$ for each function $f\in A_g\left(D\right)$ if and only if

$$\lim_{n\to\infty}T_nh_\nu\left(z\right)=h_\nu\left(z\right)$$

in $A\left(D\right)$ for $\nu=0,\,1,\,2$.

Let the sequences of positive numbers $b=\left\{\left\{b_k^{(i)}\right\}_{k=0}^{\infty}\ :\ i=\overline{0,m}\right\}$ and $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty}\ :\ i=\overline{0,m}\right\}$ satisfy the conditions (4), (5). Denote

$$H_\nu\left(z\right)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(b_k^{(i)}\right)^{\frac{\nu}{2}}g_k^{(i)}\alpha_i^k\left(z-a_i\right)^{\omega_i(k)},\nu=0,\,1,\,2. \tag{16}$$

Theorem 5. Let $T_n : A\left(D\right)\to A\left(D\right)$ be a sequence of linear k-positive operators. The sequence $T_nf\left(z\right)$ tends to $f\left(z\right)$ in $A\left(D\right)$ for each function $f\in A_g\left(D\right)$ if and only if

$$\lim_{n\to\infty}T_nH_\nu\left(z\right)=H_\nu\left(z\right)$$

in $A\left(D\right)$ for $\nu=0,\,1,\,2$.

Theorem 6. For every sequence of positive numbers $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty}\ :\ i=\overline{0,m}\right\}$ satisfying the conditions (4), (5) there exists a sequence of k-positive operators W_n : $A\left(D\right)\to A\left(D\right)$ such that the sequence of functions $W_n\left(\left(z-a_i\right)^{\omega_i(k)}\right)-\left(z-a_i\right)^{\omega_i(k)}$ converges to zero in $A\left(D\right)$ for every $i=\overline{0,m}$, $k\in Z_+$, and there exists a function $f^*\in A_g\left(D\right)$ such that

$$\lim_{n\to\infty}\left\|W_nf^*\left(z\right)-f^*\left(z\right)\right\|_{A(D),\,r_1',\ldots,\,r_m',R'}=\left\|\frac{R}{R-\left(z-a_0\right)}+\sum_{i=1}^{m}\frac{\left(z-a_i\right)}{\left(z-a_i\right)-r_i}\right\|_{A(D),\,r_1',\ldots,\,r_m',R'}>0 \tag{17}$$

for every $\max\limits_{i=\overline{1,m}}\left[\left|a_0-a_i\right|+r_i\right]<R'<R$, $r_i<r_i'<R'-\left|a_0-a_i\right|$, $i=\overline{1,m}$, $r_i'+r_j'<\left|a_i-a_j\right|$, $i,j=\overline{1,m}$, $i\neq j$.

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