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SOME PROPERTY FOR ANISOTROPIC RIESZ POTENTIAL, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR

Abstract

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator $\Delta_{B_n} = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + B_n$, $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$. The main purpose of this paper is to investigate the behaviour of Riesz potential (B_n-Riesz potential), generated by the Laplace-Bessel differential operator Δ_{B_n}. We study the anisotropic B_n-Riesz potential in the anisotropic B_n-BMO spaces.

The classical Riesz potential, an important technical tool in harmonic analysis, theory of functions and partial differential equations. The maximal function, singular integral, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_{B_n} = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + B_n, \qquad B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0$$

have been the research areas many mathematicians such as B.Muckenhoupt and E.Stein [1], I.Kipriyanov and M.Klyuchantsev [2, 3], K.Trimeche [4], L.Lyakhov [5], K.Stempak [6, 7], A.D. Gadjiev and I.A. Aliev [10, 11], I.A. Aliev and S. Bayrakci [8, 9], I.Ekincioglu, A.Serbetci [12], V.S.Guliyev [13]-[16] and others.

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator Δ_{B_n} by means of which anisotropic Riesz potential (B_n-Riesz potential) is investigated. We study the anisotropic B_n-Riesz potential in the anisotropic B_n-BMO spaces.

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$ are vectors in $\mathbb{R}^n$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \; ; \; x = (x', x_n), x_n > 0\}$, $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $\gamma > 0$, $a = (a_1, \ldots, a_n) \in (0, \infty)^n$, $|a| = \sum_{i=1}^n a_i$, $|x|_a = \left(\sum_{i=1}^n |x_i|^{2/a_i}\right)^{1/2}$, $E_+(x, r) = \{y \in R^n_+ : |x - y|_a < r\}$.

For measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\gamma = \int_E x_n^\gamma dx$, then $|E_+(0, r)|_\gamma = = \omega(n, \gamma) r^{|a| + \gamma a_n}$, where $\omega(n, \gamma) = \int_{E_+(0,1)} x_n^\gamma dx$.

Denote by T^y the generalized shift operator (B_n–shift operator) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', (x_n, y_n)_\beta\right) \sin^{\gamma - 1} \beta d\beta,$$

where $(x_n, y_n)_\beta = \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\beta}$, $C_\gamma = \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma(\gamma)\Gamma\left(\frac{1}{2}\right)}$.

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We remark that generalized shift operator T^y is closely connected with the Bessel differential expansions B_n (for example, $n=1$ see [18] and $n>1$ [19] for details).

For a fixed parameter $\gamma>0$, let $L_{p,\gamma}(\mathbb{R}^n_+)$ be the space of measurable functions on $\mathbb{R}^n_+$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

At $p=\infty$ the spaces $L_{\infty,\gamma}(\mathbb{R}^n_+)$ are defined by means of usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \underset{x\in\mathbb{R}^n_+}{esssup}|f(x)|.$$

The translation operator T^y generates the corresponding B_n-convolution

$$(f\otimes g)(x) = \int_{\mathbb{R}^n_+} f(y)[T^y f(x)] y_n^\gamma dy,$$

for which the Young inequality

$$\|f\otimes g\|_{r,\gamma} \le \|f\|_{p,\gamma}\, \|g\|_{q,\gamma}, \quad 1\le p,q,r\le\infty, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$$

holds.

Lemma 1. [11] *Let $1\le p\le\infty$. Then for all $y\in\mathbb{R}^n_+$*

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}. \tag{1}$$

Definition 1. *Let $1\le p<\infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_+)$ weak $L_{p,\gamma}$ spaces the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n_+$ with finite norms*

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r\left|\left\{x\in\mathbb{R}^n_+ \ : \ |f(x)|>r\right\}\right|_\gamma^{1/p}.$$

Definition 2. *[13] We denote by $BMO_{\gamma,a}(\mathbb{R}^n_+)$ B_n–BMO spaces the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n_+$, with finite norms*

$$\|f\|_{*,\gamma,a} = \sup_{r>0,\ x\in\mathbb{R}^n_+} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} |T^y f(x) - f_{E_+(0,r)}(x)| y^\gamma dy < \infty,$$

where $f_{E_+(0,r)}(x) = |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y f(x) y^\gamma dy$.

In [13] was investigate the $L_{p,\gamma}$ boundedness of the anisotropic B_n–maximal operator

$$M_\gamma f(x) = \sup_{r>0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y|f(x)| y_n^\gamma dy.$$

Theorem 1. *1.If $f\in L_{1,\gamma}\left(\mathbb{R}^n_+\right)$, then $M_\gamma f\in WL_{1,\gamma}(\mathbb{R}^n_+)$ and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C\|f\|_{L_{1,\gamma}},$$

where C is independent of f.

2. If $f\in L_{p,\gamma}(\mathbb{R}^n_+)$, $1<p\le\infty$, then $M_\gamma f\in L_{p,\gamma}(\mathbb{R}^n_+)$ and

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ is dependent of p,γ and n.

Consider the anisotropic B_n–Riesz potentials

$$R_\gamma^\alpha f(x) = \int_{\mathbb{R}_+^n} T^y |x|_a^{\alpha-|a|-\gamma a_n} f(y) y_n^\gamma dy, \quad 0 < \alpha < |a| + \gamma a_n$$

and the modification of anisotropic B_n–Riesz potentials

$$\widetilde{R}_\gamma^\alpha f(x) = \int_{\mathbb{R}_+^n} \left(T^y |x|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \chi_{E_+^*(0,1)_a}(y) \right) f(y) y_n^\gamma dy.$$

where $E_+^*(0,1) = \mathbb{R}_+^n \backslash E_+(0,1)$.

For anisotropic B_n–Riesz potentials the following generalized Hardy–Littlewood–Sobolev theorem is valid (see [17]).

Theorem 2. *[17] Let $0 < \alpha < |a| + \gamma a_n$.*

If $1 < p < \frac{|a|+\gamma a_n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|a|+\gamma a_n}$, $f \in L_{p,\gamma}(\mathbb{R}_+^n)$, then $I_\gamma^\alpha f \in L_{q,\gamma}(\mathbb{R}_+^n)$ and

$$\left\| R_\gamma^\alpha f \right\|_{L_{q,\gamma}} \le C_{p,\gamma} \|f\|_{L_{p,\gamma}}, \tag{2}$$

where C_p is independent of f.

If $f \in L_{1,\gamma}(\mathbb{R}_+^n)$, $\frac{1}{q} = 1 - \frac{\alpha}{|a|+\gamma a_n}$, then $R_\gamma^\alpha f \in WL_{q,\gamma}(\mathbb{R}_+^n)$ and the following inequality holds

$$\left\| R_\gamma^\alpha f \right\|_{WL_{q,\gamma}} \le C \, \|f\|_{L_{1,\gamma}}, \tag{3}$$

where C is independent of f.

Theorem 3. *Let $0 < \alpha < |a| + \gamma a_n$.*

If $1 < p < \frac{|a|+\gamma a_n}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|a|+\gamma a_n}$ is necessary for the inequality (2) to be valid.

If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{|a|+\gamma a_n}$ is necessary for the inequality (3) to holds.

Lemma 2. *Let $0 < \alpha < |a| + \gamma a_n$. Then there exists constant $C_1 > 0$ such, that*

$$\left| T^y |x|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \right| \le C_1 |y|_a^{\alpha-|a|-\gamma a_n - 1} |x|_a \tag{4}$$

for $2|x|_a \le |y|_a$.

Proof. Let's show that

$$\begin{aligned} &\left| T^y |x|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \right| \\ &\le C_\gamma \int_0^\pi \left| \left| \left(x' - y', (x_n, y_n)_\beta\right) \right|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \right| \sin^{\gamma-1} \beta d\beta. \end{aligned}$$

First estimate

$$\left| \left| \left(x' - y', (x_n, y_n)_\beta\right) \right|^{\alpha-|a|-\gamma a_n} - |y|^{\alpha-|a|-\gamma a_n} \right|.$$

By the theorem about mean value we get

$$\begin{aligned} &\left| \left| \left(x' - y', (x_n, y_n)_\beta\right) \right|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \right| \\ &\le \left| \left| \left(x' - y', (x_n, y_n)_\beta\right) \right|_a - |y| \right| \cdot \xi^{\alpha-|a|-\gamma a_n - 1} \end{aligned}$$

where

$$\min\left\{\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a,|y|_a\right\}\le\xi\le\max\left\{\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a,|y|_a\right\}.$$

Note that

$$\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a\le|x|_a+|y|_a\le\frac{3}{2}|y|_a,$$
$$\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a\ge|x-y|_a\ge|y|_a-|x|_a\ge\frac{1}{2}|y|_a.$$

and

$$\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a-|y|_a\le|x|_a+|y|_a-|y|_a\le|x|_a$$
$$|y|_a-\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a\le|y|_a-|x-y|_a\le|x|_a$$

Hence

$$\frac{1}{2}|y|_a\le\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a\le\frac{3}{2}|y|_a,$$
$$\left|\left|\left(x'-y',(x_n,y_n)_\beta\right)\right|_a-|y|_a\right|\le|x|_a$$

Thus we obtain (4). □

Theorem 4. *Let* $0<\alpha<|a|+\gamma a_n$, $p=\frac{|a|+\gamma a_n}{\alpha}$, $f\in L_{p,\gamma}(\mathbb{R}^n_+)$.
Then $\widetilde{R}^\alpha_\gamma f\in BMO_{\gamma,a}\left(\mathbb{R}^n_+\right)$ *and*

$$\left\|\widetilde{R}^\alpha_\gamma f\right\|_{BMO_{\gamma,a}}\le C\|f\|_{L_{p,\gamma}},$$

where C *is independent of* f.

Proof. Let $f\in L_{p,\gamma}\left(\mathbb{R}^n_+\right)$. For given $t>0$ we denote

$$f_1(x)=f(x)\chi_{E_+(0,2t)}(x),\quad f_2(x)=f(x)-f_1(x),$$

where $\chi_{E_+(0,2t)}$ is the characteristic function of the set $E_+(0,2t)$. Then

$$\widetilde{R}^\alpha_\gamma f(x)=F_1(x)+F_2(x).$$

where

$$F_1(x)=\int_{E_+(0,2t)}\left(T^y|x|_a^{\alpha-|a|-\gamma a_n}-|y|_a^{\alpha-|a|-\gamma a_n}\chi_{E^*_+(0,1)}(y)\right)f(y)y_n^\gamma dy$$

$$F_2(x)=\int_{\mathbb{R}^n_+\setminus E_+(0,2t)}\left(T^y|x|_a^{\alpha-|a|-\gamma a_n}-|y|_a^{\alpha-|a|-\gamma a_n}\chi_{E^*_+(0,1)}(y)\right)f(y)y_n^\gamma dy$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1=-\int\limits_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})}|y|_a^{\alpha-|a|-\gamma a_n}f(y)y_n^\gamma dy$$

is finite.

Note also that

$$F_1(x) - a_1 = \int_{E_+(0,2t)} (T^y|x|_a^{\alpha-|a|-\gamma a_n}) f(y) y_n^\gamma dy$$

$$- \int\limits_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})} |y|_a^{\alpha-|a|-\gamma a_n} f(y) y_n^\gamma dy +$$

$$+ \int\limits_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})} |y|_a^{\alpha-|a|-\gamma a_n} f(y) y_n^\gamma dy \int_{E_+(0,2t)} (T^y|x|_a^{\alpha-|a|-\gamma a_n}) f(y) y_n^\gamma dy.$$

Therefore

$$|F_1(x) - a_1| \le \int_{E_+(0,2t)} T^y|x|_a^{\alpha-|a|-\gamma a_n} |f(y)| y_n^\gamma dy$$

$$= \int_{\{y\in\mathbb{R}_+^n : T^y|x|_a < 2t\}} |y|_a^{\alpha-|a|-\gamma a_n} T^y \left|f(x)\right| y_n^\gamma dy.$$

Further, for $|x|_a < t$, $T^y|x|_a < 2t$ we have

$$|y|_a \le |x|_a + |x-y|_a \le |x|_a + T^y|x|_a < 3t.$$

Consequently

$$|F_1(x) - a_1| \le \int_{E_+(0,3t)} |y|_a^{\alpha-|a|-\gamma a_n} T^y \left|f(x)\right| y_n^\gamma dy, \tag{5}$$

if $x \in E_+(0,t)$.

By the Theorem 1 and (1), (5) for $\alpha p = |a| + \gamma a_n$

$$|E_+(0,t)|^{-1} \int_{E_+(0,t)} \left|T^z F_1(x) - a_1\right| z_n^\gamma dz$$

$$\le |E_+(0,t)|^{-1} \int_{E_+(0,t)} T^z \left|F_1(x) - a_1\right| z_n^\gamma dz$$

$$\le |E_+(0,t)|^{-1} \int_{E_+(0,t)} \left(\int_{E_+(0,3t)} |y|_a^{\alpha-|a|-\gamma a_n} T^y T^z |f(x)| y_n^\gamma dy \right) z_n^\gamma dz$$

$$\le C t^{-|a|-\gamma a_n} \cdot t^\alpha \cdot t^{(|a|+\gamma a_n)/p'} \left(\int_{E_+(0,t)} (M_\gamma(T^z|f(x)|))^p z_n^\gamma dz \right)^{1/p}$$

$$\le C_p \left\|T^z|f|\right\|_{L_{p,\gamma}(\mathbb{R}_+^n)} \le C_p \|f\|_{L_{p,\gamma}(\mathbb{R}_+^n)}. \tag{6}$$

Denote

$$a_2 = \int\limits_{E_+(0,\max\{1,2t\})\setminus E_+(0,2t)} |y|_a^{\alpha-|a|-\gamma a_n} f(y) y_n^\gamma dy.$$

Let's estimate $\left|F_2\left(x\right) - a_2\right|$.

$$|F_2(x) - a_2| \le \int_{\mathbb{R}_+^n \setminus E_+(0,2t)} |f(y)| \left| T^y|x|_a^{\alpha-|a|-\gamma a_n} - |y|_a^{\alpha-|a|-\gamma a_n} \right| y_n^\gamma dy.$$

Applying Holder's inequality we have

$$|F_2(x)-a_2| \le C|x|_a \int_{\mathbb{R}^n_+\setminus E_+(0,2t)} |f(y)||y|_a^{\alpha-|a|-\gamma a_n-1} y_n^\gamma dy \le$$

$$\le C|x|_a \, \|f\|_{L_{p,\gamma}} \left(\int_{\mathbb{R}^n_+\setminus E_+(0,2t)} |y|_a^{(\alpha-|a|-\gamma a_n-1)p'} y_n^\gamma dy \right)^{1/p'} \le$$

$$\le C|x|_a \, t^{\alpha-1-\frac{|a|+\gamma a_n}{p}} \, \|f\|_{L_{p,\gamma}} \le C|x|_a \, t^{-1} \, \|f\|_{L_{p,\gamma}}.$$

Note that for $|x|_a \le t$, $|z|_a \le 2t$, it takes place that $T^z|x|_a \le |x|_a + |z|_a \le 3t$. Thus for $\alpha p = |a| + \gamma a_n$ we obtain

$$|T^z F_2(x) - a_2| \le T^z \ |F_2(x)-a_2| \le C T^z |x|_a \, t^{-1} \, \|f\|_{L_{p,\gamma}} \le C \, \|f\|_{L_{p,\gamma}}. \quad (7)$$

Denote

$$a_f = a_1 + a_2$$

Finally, from (6) and (7) we have

$$\sup_{x,t} \frac{1}{|E_+(0,t)|_\gamma} \int_{E_+(0,t)} \left| T^y \widetilde{R}^\alpha_\gamma f(x) - a_f \right| y_n^\gamma dy \le C\|f\|_{L_p^\gamma}.$$

Thus

$$\left\| \widetilde{R}^\alpha_\gamma f \right\|_{BMO_{\gamma,a}(\mathbb{R}^n_+)} \le 2 \sup_{x,t} \frac{1}{|E_+(0,t)|_\gamma} \int_{E_+(0,t)} \left| T^y \widetilde{R}^\alpha_\gamma f(x) - a_f \right| y_n^\gamma dy \le C\|f\|_{L_{p,\gamma}}.$$

The theorem has been proved. □

Corollary 1. *Let* $p = \frac{|a|+\gamma a_n}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$.
If integral $R^\alpha_\gamma f$ *exists everywhere, then* $R^\alpha_\gamma f \in BMO_{\gamma,a}(\mathbb{R}^n_+)$ *and the inequality*

$$\left\| R^\alpha_\gamma f \right\|_{BMO_{\gamma,a}} \le C\|f\|_{L_{p,\gamma}}$$

is valid.

In the isotropic case (a.e. $a_1 = a_2 = \ldots = a_n = 1$) for the B_n–Riesz potentials

$$I^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_+} T^y |x|^{\alpha-n-\gamma} f(y) y_n^\gamma dy, \quad 0 < \alpha < n+\gamma$$

and the modification of B_n–Riesz potentials

$$\widetilde{I}^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_+} \left(T^y|x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \chi_{E^*_+(0,1)}(y) \right) f(y) y_n^\gamma dy.$$

we have the following corollaries

Corollary 2. *Let* $0 < \alpha < n+\gamma$, $p = \frac{n+\gamma}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$.

Then $\widetilde{I}^\alpha_\gamma f \in BMO_\gamma\left(\mathbb{R}^n_+\right)$ *and*

$$\left\| \widetilde{I}^\alpha_\gamma f \right\|_{BMO_\gamma} \le C\|f\|_{L_{p,\gamma}},$$

where C is independent of f.

Corollary 3. *Let $p = \frac{n+\gamma}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$.*

If integral $I^\alpha_\gamma f$ exists everywhere, then $I^\alpha_\gamma f \in BMO_\gamma(\mathbb{R}^n_+)$ and the inequality

$$\left\|I^\alpha_\gamma f\right\|_{BMO_\gamma} \leq C\|f\|_{L_{p,\gamma}}$$

is valid, where C is independent of f.

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